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WHAT HAPPENS WHEN YOU PUSH A CUBIC METER OF JELLO INTO A WORMHOLE?

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Abstract

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Inspired by the application of computer visualization of curved spaces from the perspective of an observer living within said curved space, some developments in the field of Riemannian hyperelastic mechanics are made. The defining feature of a hyperelastic material is that its behavior is defined via a stored energy function, which allows the Lagrangian formalism to be used to pose and analyze the relevant problems. To this end, a strongly typed tensor calculus formalism, inspired in part by strongly typed computer programming languages, is developed and used to develop a theory of Riemannian calculus of variations. These tools are then applied to solve several particular problems in hyperelasticity; in particular, problems involving one-dimensional hyperelastic bodies known as hyperelastic strings.

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Introduction and Motivation

The wider context for the problem addressed herein is the development of a theory for hyperelastic mechanics (a type of solid body mechanics) within Riemannian manifolds, which was originally motivated[1] by computer visualization of arbitrarily curved spaces – seeing the world from the perspective of someone living within a curved space.[2] Some applications that are of particular interest to the author are as follows.

- "exploration" of a [Riemannian] manifold from within for educational purposes by interacting directly with the topology and curvature to obtain an understanding of geodesics through direct interaction with light and motion,
- video game design incorporating non-linear spaces, especially topologically interesting features such as topological handles (which could be orientation preserving or reversing), spaces with symmetries, covering spaces, and quotient spaces, and
- digital effects for movies which includes the making of physically realistic visualizations of the theoretical spatial structure known as a wormhole.

Computer visualization is done using a generalized form of a well-known technique called raytracing, where the notion of a straight line in a vector space is replaced with a geodesic in a Riemannian manifold. This makes the visualization portion of the problem straightforward and relatively easy, although computationally expensive.

Vision is not terribly interesting if there is nothing to see. There needs to be physical objects to see within the curved space. To be clear, the words "physical" and "object" each need a definition. Define an **object**[3] in a curved space to be the image of a time-parameterized family of embeddings of a fixed compact manifold into the curved

[1]The author had originally devised this concept as an over-ambitious final project for an undergraduate computer visualization class at UCSC in 2002, though lacked a sufficient math background at the time to properly pursue it. In 2003, during the Math senior seminar, the author had learned a sufficient amount of differential geometry to implement a rudimentary version of the visualization scheme.

[2]Existing work in this type of visualization has mainly been done in the setting of spaces having constant curvature, notably http://www.geom.uiuc.edu/video/NotKnot/ – the geodesics have closed form and the high degree of symmetry allows for significant optimizations in the required calculations. Other work includes https://www.cct.lsu.edu/~werner/bh05/ which is visualization of curved spacetimes (black holes in particular), http://www.vis.uni-stuttgart.de/~dachsbcn/download/goedelengine.pdf which is visualization in general relativity, and http://www.spacetimetravel.org/wurmlochflug/wurmlochflug.html which has a fascinating video of a flight through a wormhole connecting two places on earth.

[3]This definition is specific/limited to this introduction, and is not taken from the context of the wider theory of mechanics.

space; it is a mathematical model for a the configuration of a collection of massive particles at a particular moment in time (the notion of mass requires its own definition, but that will not be made in this introduction). An object is called a **physical**[4] object if it is the solution to a particular set of physical laws[5].

From the perspective of a computer graphics programmer, the problem of creating a computer simulation of physical objects in the curved space is posed as follows. Generally there will be a large set of artist-generated geometric models – sets of triangles in $\mathbb{R}^3$ which approximate the surfaces of the models which are created in flat space – such as spaceships, humanoid forms, planets, etc. These models will play the role of the compact manifolds which will be embedded as objects into the curved space.

The programmer's typically naive approach to embedding is to map the models into the curved space by using its coordinate charts. However, because there is no structure beyond differentiability required of the coordinate charts, there is no reason that the image of the models should obey the physical laws. At the least this approach puts objects in the curved space, so that there is something to see, albeit it has no physical structure.

The real solution is to pose the physical laws as a computable process, for example, as a numerical approximation to the solution of an initial value problem. This requires starting with a mathematical model of the physical laws, picking a particular form for a solution's numerical approximation and deriving from this form an algorithm for computing the solution's numerical approximation.

This exposition will focus on the setting in which the physical laws that objects must obey are that of hyperelastic mechanics, a concept which will be detailed in Section Section 1. This choice is informed in part by the fact that traditional hyperelasticity is often posed variationally via energy functionals and that the structures and techniques involved generalize readily to the setting of curved spaces (Riemannian manifolds).

The structure of this exposition is as follows. Part I will provide an overview of traditional hyperelastic mechanics, followed by a generalization to the setting of Riemannian manifolds. In Part II, several variations of a special case of a problem in hyperelasticity involving the embedding of a 1-dimensional material into a higher-dimensional

[4]This definition is also specific/limited to this introduction.

[5]The term "physical laws" is intentionally left unspecific so that it can be specialized to whatever setting is necessary, e.g. fluid mechanics, rigid-body mechanics, elastic mechanics, etc.

space will be posed and solved. The generalization of hyperelasticity to Riemannian manifolds requires generalization of the Calculus of Variations to the setting of Riemannian manifolds, the theory of which will be developed in the latter portion of this exposition.

Part III will develop the strongly typed tensor calculus formalism which is the language used to pose the Riemannian Calculus of Variations in Part IV and the new developments of Part I. Parts III and IV are publicly available in the preprint [Dod12] and should be considered to be reference material for the developments of Parts I and II.

Part I

Overview of Hyperelasticity

1 Hyperelastic Mechanics in Euclidean Space

To provide enough context in which to frame the rest of the exposition on hyperelastic theory, some background material regarding the continuum/elastic theories set in Euclidean space involving standard vector calculus is provided. Part of the conceptual/theoretical hierarchy in the neighborhood of hyperelastic mechanics is rendered as follows, with each arrow denoting inclusion into a more general theory.

continuum mechanics	$\leftarrow$	solid mechanics	$\leftarrow$	elastic mechanics
$\uparrow$		$\uparrow$		$\uparrow$
fluid mechanics		plastic mechanics		**hyperelastic mechanics**

Continuum mechanics is the analysis of the physics of continuous bodies (i.e. not particle systems). Solid and fluid mechanics are special cases of continuum mechanics which are distinguished by a body having a preferred "rest shape" (a solid does, while a fluid does not, have a rest shape). Solid mechanics is further specialized into the study of materials which accrue permanent deformation (plastic) and materials which tend to revert to their fixed rest shape (elastic). Finally, hyperelasticity is a particular subcase of elasticity in which the elastic behavior is uniquely determined by a stored energy function.

A significant portion of the theory of continuum mechanics concerns the dynamics of 3-dimensional bodies. Let $S := \mathbb{R}^3$, denoting the **spatial manifold**. This models the space in which a body has a physical presence. The mathematical model for a continuous **body** is a 3-dimensional submanifold $M \subset \mathbb{R}^3$ (often required to be an open subset as in [MH83, pg 120], as it will be in this document) defining the body's **reference configuration** (see [Ant94, pg 414]), and the **motion** of a body (see [Ant94, pg 414]) is an orientation-preserving[6], time-parameterized family of embeddings $\phi_t \colon M \to S$ typically assumed to have as much differentiability as are used (see [Ant94, pg 414] and [HI04, pg 67]). An open subset of a body is called a **subbody**, and is a concept used in the formulation of various conservation laws. Modeled thusly, continuum mechanics is the mathematical formulation of physical laws governing the motion of such bodies.

[6]The map ϕ_t is orientation-preserving if $\det(D\phi_t) > 0$.

1.1 Conservation and Balance Laws in Continuum Mechanics

Conservation and balance laws are ubiquitous throughout physics, for example, conservation of mass, conservation of energy, balance of linear momentum, and so forth. In this setting, they are used to derive physical quantities such as mass, linear momentum, and the equations of motion. Let $\phi_t \colon M \to S$ be a motion of a 3-dimensional body. The most general conservation law typically considered in this setting is that of **conservation of mass**.

If $\rho_t \colon S \to \mathbb{R}$ is a time-parameterized family of functions such that

$$\frac{d}{dt} \int_{\phi_t(U)} \rho_t(x)\ dV(x) = 0,$$

for all subbodies $U \subset M$, then ρ_t is said to obey **conservation of mass** (see [MH83, pg 85]), and is correspondingly called a **mass density function**. The mass density function quantifies the concentration of matter throughout a motion of a body. Assuming that mass is conserved, define the **mass** of a subbody $U \subset M$ (see [MH83, pg 85]) to be

$$m(U) := \int_{\phi_t(U)} \rho_t(x)\ dV(x),$$

and there exists a **reference mass density function** (see [MH83, Theorem 5.7 and Problem 5.1]) $\rho_{\text{ref}} \colon M \to \mathbb{R}$ such that the mass of U is

$$m(U) = \int_U \rho_{\text{ref}}(X)\ dV(X).$$

It should be noted that the definition for $m(U)$ is made in spatial coordinates x, whereas the second expression for $m(U)$ (the one involving ρ_{ref}) is in terms of material coordinates X. Mass and the conservation thereof can also be posed measure-theoretically (see [Ant94, pg 428]).

Assuming that mass is conserved, and that a reference mass density function exists, define the **linear momentum** (see [Ant94, pg 430]) of a subbody $U \subset M$ to be

$$\int_U \rho_{\text{ref}}(X) \frac{\partial}{\partial t} \phi_t(X)\ dV(X).$$

Let $b_t \colon S \to \mathbb{R}^3$ be a time-parameterized family of vector-valued **body force per unit reference volume** functions, quantifying force fields that are external to the body. Let $\tau_t \colon S \times \mathbb{R}^3 \to \mathbb{R}^3$ be a time-parameterized family of vector-valued **surface traction per**

unit reference area functions[7], quantifying the internal forces of the body, where the second parameter specifies the surface normal. The **balance of linear momentum** (see [Ant94, pg 431]) is satisfied if

$$\frac{d}{dt}\int_U \rho_{\text{ref}}(X)\,\frac{\partial}{\partial t}\phi_t(X)\;dV(X) = \int_U b_t(X)\;dV(X) + \int_{\partial U} \tau_t(X, n(X))\;dA(X),$$

where $n \in \mathbb{R}^3$ denotes the outward unit normal vector field for the surface ∂U, which is assumed to be piecewise C^1. This form of the surface traction function, in particular its dependence on the pointwise value of the outward unit normal vector field for ∂U, relies on what is known as **Cauchy's postulate**[8], which follows from the satisfaction of a balance law involving angular momentum, analogous to that of linear momentum.

Satisfaction of the balance of linear momentum posits that the change in linear momentum for a subbody is exactly accounted for by the external forces on the subbody and the forces transmitted through the surface of the subbody. In the absence of body forces or surface traction, the linear momentum of a subbody is conserved.

There is an important feature of the various definitions and expressions to point out. Each of the integrals mentioned in this section can be phrased in material coordinates, as is done in [Ant94] (as an integral over a subbody $U \subset M$) or in spatial coordinates, as is done in [MH83] (as an integral over the image $\phi_t(U)$ of a subbody). An ordinary change of coordinates, involving $\det(D\phi_t)$, is what relates the two formulations.

1.2 Stress Tensor Fields

An important, and somewhat surprising, result of Cauchy, regarding the motion of a body satisfying the balance of linear momentum, is the existence of a tensor field relating the outward unit normal vector for a subbody's surface and the resulting surface traction force at that point.

Theorem 1.1 (Cauchy's Stress Theorem)**.** *If the maps $X \mapsto \tau_t(X, n)$ and $X \mapsto \rho_{ref}(X)\,\frac{\partial^2}{\partial t^2}\phi_t(X) - b_t(X)$ are continuous on M for each n, then there exists a 2-tensor field $X \mapsto T_t(X)$ such that*

$$\tau_t(X, n) = T_t(X) \cdot n.$$

[7]Here, **surface** means the boundary of any (possibly internal) surface within the body, such as the boundary of any subbody.

[8]Proved by W. Noll in 1959; see [Nol66]

See [Ant94, Theorem 7.14, pg 432] for a proof of this in material coordinates. The tensor field T_t is known as the **first Piola-Kirchhoff stress** tensor field. This tensor field is what is known as a **two-point tensor field** (see [MH83, Definition 4.14, pg 70]), in that its factors have different basepoints; $T_t(X) \in T_{\phi_t(X)}S \otimes T_X^*M$. The analogous result in spatial coordinates is proved in [MH83, Theorem 2.2, pg 134], rendering a tensor field $\sigma_t(x) \in T_xS \otimes T_x^*S$, where $x = \phi_t(X)$, and is known as the **Cauchy stress** tensor field. The Cauchy stress is not a two-point tensor field, as its factors share a common basepoint.

Assuming the balance of linear momentum holds for a body, Cauchy's Stress Theorem can be used in combination with the divergence theorem to show that

$$\int_U \rho_{\mathrm{ref}}(X) \frac{\partial^2}{\partial t^2}\phi_t(X) - b_t(X) - \mathrm{Div}(T_t(X))\, dV(X) = 0$$

for each subbody $U \subset M$, which renders the equation of motion

$$\rho_{\mathrm{ref}} \frac{\partial^2}{\partial t^2}\phi_t = b_t + \mathrm{Div}\, T_t,$$

assuming that the integrand is continuous. Thus, for a fixed choice of ρ_{ref} and b_t, the dynamics of a continuous body is determined by its stress tensor field.

The two stress tensors mentioned so far are related by what is known as the **Piola transformation**, which can be thought of as "pushing" or "pulling" factors of tensor fields back and forth between the spatial and material formulations (see [MH83, Section 1.7]). With the assumption that ϕ_t is an embedding, this process is invertible, and essentially amounts to a coordinate transformation on a single factor of a tensor field, via ϕ_t. With $F_t(X) := D\phi_t(X) \in T_{\phi_t(X)}S \otimes T_X^*M$ denoting the **deformation gradient** and $J_t(X) := \det(F_t(X))$, the relationship is

$$T_t(X)^{ij}\, F_t(X)_j^k = J_t(X)\, \sigma_t^{ik}(\phi_t(X)).$$

This can also be expressed in terms of $F_t^{-1}(X) \in T_XM \otimes T_{\phi_t(X)}^*S$, as is done in [MH83, pg 135]. There is another tensor field, called the **second Piola-Kirchhoff stress** tensor field $S_t \in T_XM \otimes T_X^*M$, which is the first Piola-Kirchhoff stress with its other factor pulled back to the material (see [MH83, Definition 2.8, pg 136]).

Because the Cauchy stress, and first and second Piola-Kirchhoff stress tensor fields are related by an invertible transformation, they encode equivalent information – how internal stresses are propagated through the body.

1.3 Constitutive Equations

The different, equivalent stress tensor fields encode the force-response behavior of the body under deformation. What defines these stress tensor fields, thereby defining the modeled continuous material, is known as a **constitutive equation**. A completely detailed general definition will not be made here, as the full generality is not used in this exposition, but the concept is that a constitutive equation is a map from the space of histories of motions of a body to the space of stress tensor fields (see [MH83, pg 182] and [Ant94, pg 446]). The use of the entire history of the motion of a body is to allow for materials that accrue permanent deformation based on past deformation (e.g. plastic materials), or rate of deformation (e.g. non-Newtonian fluids). A constitutive equation is not required to depend on the entire history of the body. Materials defined by constitutive equations which have simpler dependencies are the subject of the various subclasses of continuum mechanics.

1.4 Elasticity

There is a particular subclass of continuous materials, known as **elastic materials**, whose constitutive equations have a particular form; they depend only on the deformation gradient $F_t := D\phi_t$. In more detail, the functional dependence of the first Piola-Kirchhoff stress tensor field $T_t(X)$ is

$$T_t(X) = \hat{T}(F_t(X), X),$$

for some function $\hat{T}$ of the deformation gradient F_t and the material coordinate (see [Ant94, pg 447]). The other forms of the stress tensor (e.g. Cauchy stress) have qualitatively analogous formulations.

Another, more pure-geometric treatment of elasticity (framed in the somewhat more general setting of **thermoelasticity**, which includes the thermodynamical properties of the material), approaches the constitutive equation from the notion of **locality**, in which the dependence on the body's motion of the stress tensor field at each point is only through the body's configuration in an arbitrarily small neighborhood around said point (see [MH83, pg 189]). This postulate, along with a inequality involving **entropy production** (see [Ant94, pg 476] and [MH83, pg 190]) which relates the temperature and deformation rate of a body with its constitutive function, are the hypotheses for a theorem of Coleman and Noll in (1963), stating that the constitutive function $T_t(X)$ depends only on $F_t(X)$, X, and the temperature at X.

It should be noted that certain elastic materials do not obey locality, such as **incompressible** elastic materials (see [MH83, pg 189]) (defined by $J_t = 1$); the incompressibility manifests via the presence of a **pressure** function in the equations of motion, which is what effectively enforces the constraint[9].

To put concisely, a material (discounting temperature) is elastic if and only if its behavior under stress depends at each material point only on that point and the body's local deformation (i.e. deformation gradient) at that point (see [Ant94, pg 447, 477]). Another interesting way to look at elasticity this is that an elastic material is one in which energy can not be dissipated (see [Riv97, pg 325]). Finally, the most simplistic definition of an elastic body is one that returns to its original shape when external forces are removed (see [HI04, pg 1]).

There are two important classes of elastic materials which model real world materials that render simplifications to posing and solving problems in elastic mechanics. The classes discussed in this exposition are isotropic materials, whose stress response to deformation is equivariant under local rotations, and homogeneous materials, whose stress tensor fields don't depend explicitly upon the material point.

Put more precisely, an elastic material with constitutive equation $\hat{T}(F, X)$ is **homogeneous** if $\frac{\partial \hat{T}}{\partial X} = 0$ (see [MH83, pg 191]).

To give a rigorous definition of an isotropic elastic material, it is necessary to first introduce the concept of **material frame indifference** (see [MH83, pg 194]), which effectively states that the stress tensor is invariant under isometric spatial transformations. The result of this is that the stress tensor depends on only X and

$$C_t(X) := F_t^T(X) F_t(X) \in T_X M \otimes T_X^* M,$$

a quantity known as the **right Cauchy-Green deformation tensor**, and therefore that the constitutive equation can be written as a function of X and C_t (see [Ant94, pg 416]);

$$T_t(X) = \overline{T}(C_t(X), X),$$

[9] In the setting of hyperelasticity, the pressure function acts as a Lagrange multiplier for the problem which is posed variationally; see [MH83, pg 279].

where $\overline{T}(C,X)$ is the unique map that factors through $\hat{T}(F,X)$ with respect to $C := F^T F$, i.e. $\overline{T}\left(F^T F, X\right) = \hat{T}(F,X)$.

An elastic material satisfying material frame indifference, with constitutive equation $\overline{T}(C,X)$, is **isotropic** (see [MH83, pg 220]) if $\overline{T}\left(R^T C R, X\right) = \overline{T}(C,X)$ for all orientation-preserving rotations $R \in T_X M \otimes T_X^* M$.

1.5 Hyperelasticity

The subclass **hyperelastic mechanics** of elastic mechanics is defined simply by the existence of a stored energy function whose derivative renders the stress tensor (see [Ant94, pg 447]). In more detail, a **stored energy function** for a hyperelastic material (see [MH83, pg 210]) is a real-valued function $U(X,F)$ such that

$$T_t(X) = \frac{\partial U}{\partial F}\left(X, D\phi_t(X)\right).$$

Note that because the F parameter is 2-tensor-valued, the derivative $\frac{\partial U}{\partial F}$ is 2-tensor-valued.

The existence of a stored energy function allows certain problems in elastic mechanic to be posed variationally, the solutions to the equations of motion being critical points of an energy functional. The extra structure allows the use of additional tools in functional analysis to be used to attack the problem, for example in certain cases establishing the existence and uniqueness of a critical point. The formulations in this section are all within the standard theory of the calculus of variations.

Let

$$L(X,x,F,v) := \frac{1}{2}\rho_{\text{ref}}(X)\,|v|^2 - U(X,F),$$

called the **Lagrangian**, and define energy functional

$$\mathcal{L}(\phi_t) := \int_M L\left(X, \phi_t(X), D\phi_t(X), \frac{\partial}{\partial t}\phi_t(X)\right) dV(X). \tag{1.1}$$

The **Euler-Lagrange equation** for this functional (see [MH83, pg 277]) is

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial v}\left(X, \phi_t(X), D\phi_t(X), \frac{d}{dt}\phi_t(X)\right)\right) \\ &= \frac{\partial L}{\partial x}\left(X, \phi_t(X), D\phi_t(X), \frac{d}{dt}\phi_t(X)\right) \\ &\quad - \operatorname{Div}\left(\frac{\partial L}{\partial F}\left(X, \phi_t(X), D\phi_t(X), \frac{d}{dt}\phi_t(X)\right)\right), \end{aligned}$$

which can be simplified due to the particular form of L;

$$\frac{\partial L}{\partial v}(X, x, F, v) = \rho_{\text{ref}}(X)\, v,$$
$$\frac{\partial L}{\partial x}(X, x, F, v) = 0,$$
$$\frac{\partial L}{\partial F}(X, x, F, v) = -\frac{\partial U}{\partial F}(X, F),$$

so the Euler-Lagrange equation becomes

$$\begin{aligned}\rho_{\text{ref}}(X)\frac{\partial^2}{\partial t^2}\phi_t(X) &= \operatorname{Div}\left(\frac{\partial U}{\partial F}(X, D\phi_t(X))\right) \\ &= \operatorname{Div}(T_t(X)).\end{aligned}$$

Put more succinctly, $\rho_{\text{ref}}\frac{\partial^2}{\partial t^2}\phi_t = \operatorname{Div} T_t$, which is the equation of motion derived in Section 1.2, assuming zero external forces.

If the external force field is conservative, i.e. it has a potential energy function $P\colon S \to \mathbb{R}$ such that $\operatorname{Grad} P$ is the force field, then the equation of motion can be derived variationally. Define the Lagrangian

$$L(X, x, F, v) := \frac{1}{2}\rho_{\text{ref}}(X)|v|^2 - (U(X, F) + P(x))$$

with energy functional defined as in (1.1), the Euler-Lagrange equation is

$$\rho_{\text{ref}}\frac{\partial^2}{\partial t^2}\phi_t = \operatorname{Grad} P + \operatorname{Div} T_t.$$

1.6 Hyperelastic Materials With Symmetries

In the hyperelastic setting, homogeneous materials and isotropic materials have particularly simple definitions. If $\overline{U}(X, F)$ is the stored energy function for a hyperelastic material, then that material is homogeneous if $\frac{\partial \overline{U}}{\partial X} = 0$. To formulate the condition of isotropy, it is necessary to assume material frame indifference to show that the function $\overline{U}$ depends on F only through $C := F^T F$, as was done in Section 1.4, so that there is a unique function $U(C, X)$ that factors through $\overline{U}(F, X)$ with respect to C, i.e. $U\left(F^T F, X\right) = \overline{U}(F, X)$. Then the condition of $\overline{U}$ representing an isotropic material is $U\left(R^T C R, X\right) = \overline{U}(C, X)$ for all orientation-preserving rotations $R \in T_X M \otimes T_X^* M$.

An important property of an isotropic elastic material's stored energy function requires the introduction of some natural functions in linear algebra. Let $n := \dim M$.

For $k \in \{1, \ldots, n\}$, let

$$\sigma_k \colon TM \otimes T^*M \to \mathbb{R}$$

denote the **kth tensor invariant** function; this is the kth symmetric polynomial in the eigenvalues of its argument. These functions are defined as the sign-alternating coefficients of the characteristic polynomial of the argument;

$$\mathrm{CharPoly}_C(x) = x^n - \sigma_1(C)\, x^{n-1} + \sigma_2(C)\, x^{n-2} + \cdots + (-1)^{n-1}\sigma_{n-1}(C)\, x + (-1)^n \sigma_n(C).$$

The first and last invariants are ubiquitous; $\sigma_1(C) = \mathrm{Tr}(C)$ and $\sigma_n(C) = \det(C)$. It should be noted that each tensor invariant is rotationally invariant, i.e. $\sigma_k\left(R^T C R\right) = \sigma_k(C)$ for all orthogonal transformations R.

The kth tensor invariant is a function which measures [$\binom{n}{k}$ times the mean] local change in k-volume. Here, 1-volume is length, 2-volume is area, and n-volume is full-rank volume (in the sense of a volume form on the manifold). The isotropy condition implies that the function U only depends on $\sigma_1(C), \ldots, \sigma_n(C), X$ (see [MH83, Proposition 5.10, pg 221]), and therefore uniquely factors through a function $\tilde{U}(\sigma_1(C), \ldots, \sigma_n(C), X)$. Intuitively, it makes sense that an isotropic material's stored energy function should only depend on the tensor invariants, as they are measuring all possible ways that the material can be stretched locally.

An **incompressible material** is one having no local change in volume, which is expressed via tensor invariants as requiring that $\sigma_n(C) = 1$ (i.e. $\det(C) = 1$). In this case, $\tilde{U}$ is only a function of X and the first $n-1$ tensor invariants of C.

2 Hyperelastic Mechanics in Manifolds

This section will provide an overview of some existing results in the generalization of hyperelasticity to differentiable and Riemannian manifolds. The standard theory of continuum mechanics, posed with the assumption that physical space is a Euclidean [vector] space, is formulated in a way that takes advantage of the high degree of structure present in said Euclidean space. Furthermore, these special case structures are used to make certain identifications of spaces and represent most quantities using a single vector type. In the setting of manifolds, these identifications are neither possible nor desirable. However, embracing the multitude of distinguishable types[10] in the set-

[10] The word **type** in this context is synonymous with the word **space**; a vector space V and a different vector space W are distinguishable types, as is a manifold M.

ting of differentiable manifold theory is not only necessary, but advantageous. It allows for greater insights into what certain objects actually are and how they can be used together, even sometimes suggesting how they should be used. This "strong typing" is a key point made in Part III. The treatment of continuum and elastic mechanics using the language and theory of Riemannian manifolds was largely initiated by Marsden and Hughes in the mid-1980s (see [MH83] and [SH97, pg 327]). One of the critical advantages to this approach was to make some progress in the direction of a more strongly typed formulation of the theory. Other modern treatments of continuum mechanics such as [SH97, EM70, Eps10] also use these type distinctions to great benefit.

2.1 Special Structures of Euclidean Space Unavailable in General Manifolds

This section details certain special structures of Euclidean space used in the formulation of traditional continuum mechanics for which alternative formulations must be derived to make a generalization to the setting of Riemannian manifolds. For the purposes of this section, let V denote the Euclidean space which physical space is modeled upon.

- The vector space V has a globally trivializable tangent bundle TV, canonically isomorphic to $V \times V$, giving canonical base and fiber component projections. This allows positions and directions to be decoupled and dealt with separately, which is a feature that drastically simplifies the mathematics necessary to pose the relevant theory.

- The global trivialization of TV allows the definition of the linear-map-valued differential; if $f \colon U \to V$, where $U \subset V$, then $Df \colon U \to \operatorname{Hom}(V, V)$ gives all 1st order derivative information for f without explicitly including f itself; the deformation gradient is the prime instance of this structure in continuum mechanics.

- The inner product on a Euclidean space, which is a linear isometry $V \to V^*$, admits a structure-respecting identification of V with V^* (and therefore TV with T^*V). This allows the distinction between vectors and covectors to be eliminated.

These features essentially allow the full theory to be posed entirely using the single vector type V; positions, velocities, and the linear momenta of subbodies are each represented as V type vectors. While convenient from one perspective, discarding the distinction between the three makes it less clear what each quantity actually is and how they relate to each other on a deeper level. This makes more laborious the process of generalizing to the setting in which these identifications are impossible.

Discarding the assumption that the physical space is a Euclidean space, and requiring only that it is a Riemannian manifold, it is both necessary and useful to restore the type distinctions regarding each of the various physical quantities. Let (M, g_M) and (S, g_S) denote the material and spatial manifolds, and let $\phi\colon M \to S$. The failure of some of the special features listed above are handled as follows.

- Because TS is not necessarily globally trivializable, position and velocity quantities are not separable in the same way they are in the Euclidean setting. The consequences of this are that a velocity vector must also encode its positional base point and that there is no direct generalization of the differential of an S-valued map; the differential must take values in T^*S.

- The tangent map $T\phi\colon TM \to TS$ encodes all 0th and 1st order derivative information (0th order being the base map ϕ itself) and should be thought of as an inseparable object. This is the appropriate generalization of the deformation gradient from the Euclidean setting to the Riemannian setting (see [SH97, pg 332]).

- A direct generalization of linear (or angular) momentum of a subbody requires the use of a nondiscrete Lie group action on (S, g_S) by isometries which is a geometric structure which quantifies symmetries of a Riemannian manifold. However, not all Riemannian manifolds have symmetries in this sense, so in general, there is no nondiscrete Lie group action on a Riemannian manifold. If such an action does exist, then there is an induced vector-valued form on T^*S which is known as the **momentum map** of the action (see [MR99, pg 367]). The momentum map allows a definition of the momentum of a subbody of M and therefore provides a means to generalize the notion of balance laws and derive equations of motion. The Lie group action may not be transitive, meaning that the vector-valued momentum may have a dimension less than that of the manifold S, which brings into question the uniqueness of solutions to the corresponding equations of motion.

The momentum-map-based formulation of subbody momentum and corresponding equations of motion will not be addressed in this exposition because it is not used in hyperelasticity. The notion of a balance law is not addressed in this section, as it requires more machinery to properly formulate than can be mentioned in a single paragraph.

2.2 Conservation and Balance Laws in Continuum Mechanics on Manifolds

In the traditional setting of continuum mechanics, conserved quantities and balance laws are formulated in terms of integrals of quantities over the subbodies of a region of space. The proper generalization to the setting of manifolds – not even requiring a Riemannian structure – uses the language of differential forms and the integrals thereof.

Conservation and balance laws deal with what are known as **extensive properties** which are each defined as an integral of a **density** (see [Eps10, pg 170]). In mathematical language, an extensive property is a measure defined on the body M, and the density is a differential k-form, where $k = \dim M$. The density measures the concentration of the extensive property, per unit k-volume, in the body. It is important to note that an extensive property is a scalar function. For vector-valued functions (such as the linear or angular momentum in the Euclidean setting), more machinery is required – in particular, vector-valued differential forms, such as the momentum map induced by a Lie group action.

Conceptually, a balance law is simply a statement that the rate of change in an extensive property over a subbody can only be due to the rate of the creation/destruction of said extensive property in said subbody, along with the flux of that said extensive property across said subbody's boundary (see [Eps10, pg 171]). Let P_t denote a time-parameterized extensive property function with time-parameterized density

$$\rho_t \in \Gamma\left(\bigwedge^k T^*M\right),$$

so that

$$P_t\left(U\right) := \int_U \rho_t$$

is the value of the extensive property for subbody $U \subset M$ at time $t \in \mathbb{R}$. If

$$s_t \in \Gamma\left(\bigwedge^k T^*M\right)$$

is the time-parameterized source density at time t (giving the rate of creation/destruction of said extensive property in the body), and

$$\tau_t^U \in \Gamma\left(\bigwedge^{k-1} T^*M\right)$$

is the time-parameterized flux density for each subbody U at time t, defined such that the flux of $P_t(U)$ is $\int_{\partial U} \tau_t^U$, then the corresponding balance law has the form

$$\int_U \frac{\partial \rho}{\partial t} = \int_U s_t - \int_{\partial U} \tau_t^U,$$

and with the assumption of enough regularity of the function $(m, t) \mapsto \rho_t(m)$, the time derivative can be passed outside of the integral, and the balance law for the extensive property P_t has the form

$$\frac{d}{dt} P_t(U) = \int_U s_t - \int_{\partial U} \tau_t^U \tag{2.1}$$

for all subbodies $U \subset M$. A conservation law is simply the balance law (2.1) with the right-hand side identically equal to zero for all $U \subset M$ (see [Eps10, pg 172]).

2.3 Deformation/Stress Tensor Fields and Elasticity

This section will review some of the concepts of continuum/elastic mechanics as discussed in Section Section 1, and then briefly detail their generalizations into the setting of Riemannian manifolds. Using the tools of the "strongly typed" formalism commonly used in differential geometry, such as distinguishing a vector space and its dual, these constructions come quite naturally.

The notion of a deformation tensor field (often referred to simply as a deformation tensor), of which the deformation gradient $T\phi \colon TM \to TS$ is the prime example, is used in quantifying the local deformation of a body embedding. A stress tensor field (often referred to simply as a stress tensor) is, in the context of continuum mechanics, a function of the body embedding's history and is what quantifies the body's internal forces based on the configuration of the body. Knowledge of the stress tensor is sufficient to determine the statics/dynamics of a continuous body. The main examples of stress tensors are the first and second Piola-Kirchhoff and the Cauchy stress tensors, each of which quantify stresses in differently formulated but informationally equivalent ways. An elastic body is one whose stress tensor, at a given point, depends on the body configuration only through the spatial derivative components of the deformation tensor at the given point (i.e. not on time derivatives or any of the past of the body's configuration).

Let $S(m,F)$ denote the stress tensor for an elastic material (which could be any of the stress tensor fields mentioned above), where the type of the argument F is $F\colon T_mM \to T_sS$ for some $s \in S$. The notion of material frame indifference, discussed in Section 1.4, was addressed in the generality of Riemannian manifolds (see [MH83, pg 194]) and leads to the result that the stress tensor $S(m,F)$ depends on F only through the quantity $C := F^T \circ F$, where $F^T\colon T_sS \to T_mM$ is defined by the metric structures on (M, g_M) and (S, g_S) by the equality $g_S \circ F = g_M \circ F^T$. Using the inverse metric tensor $g_S^{-1}\colon T^*S \to TS$ and the natural adjoint $F^*\colon T_s^*S \to T_m^*M$, an explicit formula for the transpose of F can be attained;

$$F^T = g_M \circ F^* \circ g_S^{-1} |_{T_sS} \, .$$

Therefore, under the assumption that the stress tensor field is materially frame indifferent, there is a unique function $\overline{S}(m,C)$, where $C\colon T_mM \to T_mM$, such that $\overline{S}\left(m, F^T \circ F\right) = S(m,F)$. Recalling that $T\phi\colon TM \to TS$ is the deformation gradient, which is the most basic deformation tensor for elasticity, the quantity

$$C := (T\phi)^T \circ T\phi\colon TM \to TM$$

is sufficient for use as a deformation tensor. The field C is the generalized version of the right Cauchy-Green deformation tensor field (see [SH97, pg 334] and [JEMW01, pg 17]). The fact that C is a vector bundle morphism over the identity map lends itself to the definition of an isotropic material and the quantification of isotropic materials through their tensor invariant functions. This will be discussed in Section Section 3.

It should be noted that the functional notation $S(m,F)$ is somewhat abusive, because the type of F depends on the value of m. This abuse is frequently in the relevant literature, notably [MH83]. This caveat will be removed in Section Section 3 with a reformulation of such functions using a concept which extends the notion of two-point tensor fields seen in [MH83, pg 48, 70].

3 Developments in Riemannian Hyperelasticity

This section will provide an outline and discussion of the main developments of this exposition, which is summarized as follows.

- An overview of the development of a strongly typed formalism for coordinate-free tensor calculus, which is detailed in III,
- an overview of the development of a theory of Riemannian calculus of variations in said formalism, which is detailed in IV,
- the application of both to the theory of hyperelastic mechanics in Riemannian manifolds (in this section), and
- solutions to particular problems in hyperelastic mechanics, detailed in Part II.

It should be noted that the strongly typed tensor calculus formalism detailed in Part III will be used in this exposition, from here onward.

3.1 Strongly Typed Tensor Calculus Formalism

The strongly typed tensor calculus formalism detailed in Part III should be thought of as a framework for doing coordinate-free calculus in the setting of differential geometry, utilizing a small set of constructions (e.g. tensor bundles, vector bundle splittings, and most crucially, pullback bundles) to regularize and enrich the set of available types with which to distinguish objects. This has the aim of clarifying complex calculations and reducing human error. It has the advantages of not obscuring the geometry of the problem with artificial structures such as choices of coordinates, using the type distinctions to suggest what each object is or how it could be used, and having enough regularity to be appropriate for a computerized system for symbolic calculations (a future direction for this research). It is **important to note** that from this point on, the material detailed in Part III will be used freely, and thus the reader is advised to read Part III before proceeding.

The pullback bundle aspect of the strongly typed formalism was primarily inspired by [Xin96] in its treatment of the theory of harmonic maps between Riemannian manifolds. Pullback bundles fit nicely into covariant calculus because the pullback of a bundle having a covariant derivative has a naturally induced covariant derivative that is defined uniquely by what is effectively the chain rule.

Linear maps and fields thereof are conveniently represented in tensor bundles using the canonical isomorphism $\operatorname{Hom}(V, W) \cong W \otimes V^*$, where V and W are vector bundles over the same manifold. This generalizes slightly by allowing tensor products of bundles having different base spaces, the archetypical example being $TS \otimes T^*M \to S \times M$, which is a vector bundle sufficient for quantifying the total first derivative of maps of the form $M \to S$, which then take the form of a tensor field and to which the tools of covariant calculus can be applied. In particular, the tangent map $T\phi \colon TM \to TS$ of ϕ takes the form of a tensor field as

$$\nabla \phi \in \Gamma_{\phi \times_M \mathrm{Id}_M} (TS \otimes T^*M) \,,$$

or the often-useful type refinement

$$\overline{\nabla \phi} \in \Gamma (\phi^* TS \otimes T^*M) \,.$$

Here, the symbol ∇ denotes the operator which produces the tangent map as a tensor field (see Section 20). While not directly inspired by outside sources, the tensor product of vector bundles over different base manifolds occurs in [KMS93, MH83].

The vector bundle splitting technique was inspired by the use of **partial functional derivatives** in [MR99, pg 76] in its treatment of Hamilton's equations, though the splitting technique occurs in a more directly equivalent way in [MH83, pg 276]. The idea of splitting a vector bundle on which a covariant derivative is defined led to the notion of **partial covariant derivatives** (**PCD**s), which essentially give the component of a covariant derivative along a particular subbundle or distribution. This led to techniques of calculation which often feel like proofs involving naturality properties, such as "the particular PCD of this tensor field is zero, showing that it is horizontal with respect to the distribution" or "this particular PCD is the identity tensor on this vector bundle".

Finally, it is **important to note** that the pullback in the strongly typed formalism (which coincides with the categorical definition) differs slightly from the standard notion of pullback in manifolds theory. The specifics are as follows. Let M and N be manifolds, let $f \colon M \to N$, and let $\alpha \in \Gamma (T^*N)$. The standard notion of pullback is defined as "precomposition with pushforward", meaning that

$$f^* \alpha = \alpha \circ Tf \in \Gamma (T^*M) \,.$$

However, in the strongly typed formalism, the pullback just induces a field on the pullback bundle. In this scenario,

$$f^*\alpha \in \Gamma\left(f^*T^*N\right).$$

The standard notion of pullback can be expressed within the strongly typed formalism as

$$\begin{array}{ccc} \text{standard} & & \text{strongly typed} \\ f^*\alpha & \leftrightarrow & f^*\alpha \cdot_{f^*T^*N} \overline{\nabla f}, \end{array}$$

where $\overline{\nabla f} \in \Gamma\left(f^*TN \otimes T^*M\right)$ is the tensor field expression for the tangent map of f. See Section 19 for full detail on the pullback formalism.

See Section 13 for the notation and conventions of the strongly typed tensor formalism.

3.2 Riemannian Calculus of Variations

The prime motivation for the development of a theory of covariant calculus of variations was to pose problems in hyperelastic mechanics in the context of Riemannian manifolds (in which covariant differentiation is a natural structure) and apply variational tools to obtain their solutions. The lack of a complete generalization of the standard theory of calculus of variations to the covariant/Riemannian setting, in particular one that allowed Lagrangians defined arbitrarily on the first jet space of a map of the form $M \to S$, was the first hurdle. There were existing similar developments, notably [Nis02] in its development of the first and second variations of the energy functional for harmonic maps, [And92] which discusses an approach using exterior calculus, and [MH83, pg 277].

The variational formulation made in [MH83] deserves special attention, as it is most similar to the developments made in this exposition. In particular, it uses a natural splitting of the first jet bundle of time-parameterized maps of the form $M \to S$, and defines the energy functional using a Lagrangian defined on this splitting. The first variation of the energy functional is derived in a rather formal and undetailed way, using "partial derivatives" analogous to the PCD described in 22.4, eventually obtaining an Euler-Lagrange equation. It should be noted that no covariant structure is used in the bundle splitting or in the derivation of the first variation, so while it is qualitatively similar to the theory developed in IV, it is inequivalent. The types of the various constructions, such as the "partial derivatives" and the divergence used in the Euler-Lagrange equation, are frustratingly not detailed or discussed.

The Riemannian calculus of variations developed in IV gives direct analogs of the results standard calculus of variations theory, where PCDs replace the coordinate-based partial derivatives of the standard theory. In particular, the formalism defines an energy functional $\mathcal{L}$ on maps of the form $\phi\colon M \to S$, whose Lagrangian depends on the first jet of said maps – in particular, the Lagrangian is a real-valued function on the tensor bundle $TS \otimes T^*M \to S \times M$;

$$\mathcal{L}(\phi) := \int_M L \circ \widetilde{\nabla}\, \phi \, dV_{(M,g_M)}.$$

First and second variations are derived, along with Euler-Lagrange equations, as well as a conserved quantity which is the direct analog of the total energy of a simple mechanical system. The only qualitative difference in these developments is the presence of the Riemannian curvature tensor for ∇^{TS} in the second variation which is due to the commutation of a particular pair of covariant derivatives.

The complexity of the calculations to derive the first and second variations are part of what motivated the development of the strongly typed tensor calculus formalism. Coordinate-based calculations became too unwieldy and error-prone, often obscuring the geometry of the problem due to their heavily index-based notation. Even limited to the energy functional which defines a harmonic map between Riemannian manifolds, as is done in [Nis02] and [Xin96], the coordinate-based derivations are rather difficult even though the harmonic-map-defining functional could be considered to be the simplest nontrivial such functional.

Use of the strongly typed tensor calculus formalism makes the necessary calculations more regular and mechanical. This fact lends itself to a computer implementation of said calculations, making feasible the design of computer systems for automation of some portions of these computations, human-guided symbolic computation, and automated proof-checkers for the sequences of equalities resulting from such computations. This will be a subject of interest in the author's future work.

3.3 Applications to Hyperelastic Mechanics in Riemannian Manifolds

This section provides an overview of hyperelasticity as developed in this exposition, with references to further details in later sections.

3.3.1 Material Manifold

As in the existing theory of continuum mechanics in Riemannian manifolds, let (S, g_S) denote the spatial manifold; the manifold used to represent physical space. The standard way to think about the material manifold for a continuous body is as an embedded submanifold $M \subset S$, giving the rest state of the material body. This is certainly a valid generalization of the Euclidean theory, but there is a subtle complication in doing this. The spatial manifold S is not necessarily a homogeneous manifold, meaning that the pointwise curvature of S may depend on the point in S considered, and therefore the rest state of a body represented as the submanifold M would be inherently tied to that particular embedding. It may be the case that that particular location is the only place in which the body can experience no internal stress forces due to the curvature of space. Furthermore, requiring that M be an embedded submanifold of S does not address one of the motivating applications – embedding "flat $\mathbb{R}^3$" bodies into curved space, as the curved space may not possess any "flat" portions at all, thereby making a zero-internal-stress embedding entirely impossible. For example, embedding a solid cube (or the humanoid form of an astronaut, etc.) from flat $\mathbb{R}^3$ into a hyperbolic space.

Thus the material manifold will be declared to be an arbitrary, compact Riemannian manifold (M, g_M), whose shape (topology, size, curvature) determines the shape of the body that is to be embedded. The body embedding is still a map $\phi\colon M \to S$. However the rest state is abstracted to the identity map $\mathrm{Id}_M\colon M \to M$, which is an isometry. In the special case where M is chosen to be an isometrically embedded submanifold of S, the rest state is the inclusion, which coincides with the traditional definition. Allowing M to be a manifold that is not necessarily embedded in S allows the dynamics of "non-native" bodies (in the sense that their rest state does not come from the physical space S) to be modeled, such as a "natively flat" body in hyperbolic space, or a "natively hyperbolic" body in a non-uniformly curved space. From the perspective of video game or movie effect design, this would be used to model a spaceship from a flat universe that has, through the magic of science fiction, been teleported into a hyperbolically curved universe, in which the spaceship has no rest state – there would always be internal stresses due to the curvature of space.

The validity of this particular abstraction will now be addressed. The key concept is that the rest state of the material body is being decoupled from the space in which it will be embedded, and that the material manifold M (along with its metric g_M)

defines the rest state. The material manifold could be thought of as a submanifold of the "native space" from which the body comes and may have nothing to do with the physical space in which it will be embedded. A physically realistic example of this in Euclidean 2-space is described as follows. Take 6 springs, all having the same length, and assume that the springs can only compress/tense in a linear fashion (e.g. a shock absorber), and that the joints at which they will connect are incompressible. Connect 3 of them at their ends to form a triangle. connect to each of the 3 vertices of the triangle one of the 3 remaining springs, each pointing inward. Connect the inward-pointing springs in the center of the triangle. This will cause the outside edge springs to be under tension and the inward-pointing springs to be under compression (i.e. the body has internal stresses), and is therefore not a rest configuration. Furthermore, it is impossible for this body to attain a rest configuration in the physical space at all, as this would require all the vertices (including the central one where the 3 inward-pointing springs connect) to be equidistant. However that is equivalent to the geometric problem of isometrically embedding a tetrahedron in Euclidean 2-space – something which is impossible. The material manifold for this construction would be a flat 2-dimensional manifold (flat in the sense that its curvature tensor is identically zero) having an isometric embedding into Euclidean 3-space.

There is precedent in existing literature for the use of an abstract material manifold, notably [Eps10, pg 23], who considers the collection of diffeomorphisms of a material manifold onto itself, without reference to the spatial manifold at all. The notion of a **material virtual displacement** is used to quantify changes in the material itself, which suggests that a diffeomorphism of the material manifold onto itself should be called a **material displacement**. In [SH97, pg 332], the **placement** of a material body at time t was defined to have the form $\chi_t \colon B \to E$, where B is the material manifold and E is a Euclidean space. The **reference placement** of the body is then defined to be $\chi_0(B) \subset E$, and the composition $\chi_t \circ \chi_0^{-1}$ defines a referential description of the motion of the body. The distinction between the body manifold B and the reference placement $\chi_0(B)$ suggests that B is not required to be an embedded submanifold of E.

Finally, there is nothing about the mathematical formulation of the theory of continuum/[hyper]elastic mechanics in Riemannian manifolds that requires that the material manifold be an embedded submanifold of the spatial manifold, so that restriction will not be made.

3.3.2 Deformation Gradient

Existing theory has already established that if the material embedding is a map $\phi\colon M \to S$, then its tangent map $T\phi\colon TM \to TS$ is the correct formulation for the deformation gradient in this context. In the strongly typed tensor formalism, in which the Riemannian calculus of variations is developed (see IV), this is the quantity

$$\nabla \phi \in \Gamma_{\overline{\phi}}\left(TS \otimes T^*M\right),$$

recalling that this means that $\nabla \phi\colon M \to TS \otimes T^*M$ such that $\pi^{TS\otimes T^*M}_{S\times M} \circ \nabla \phi = \overline{\phi}$, where $\overline{\phi} := \phi \times_M \mathrm{Id}_M$ is a type refinement of the base map ϕ. The type refinement

$$\overline{\nabla \phi} \in \Gamma\left(\overline{\phi}^* \left(TS \otimes T^*M\right)\right) \cong \Gamma\left(\phi^*TS \otimes T^*M\right)$$

is also useful, as it is a section of the tensor bundle $\phi^*TS \otimes T^*M \to M$, on which there are covariant derivatives induced from the Levi-Civita connections on TS and TM. The quantities $\nabla \phi$ and $\overline{\nabla \phi}$ are equivalent and each one can be recovered from the other. Which one to use depends on the context of the particular formulation. The essential property of $\nabla \phi$ is that it contains all 0^{th} and 1^{st} order derivative information for the map ϕ, in the form of a tensor field upon which covariant derivatives are defined.

3.3.3 Right Cauchy-Green Deformation Tensor Field

In the strongly typed formalism, the right Cauchy-Green tensor field takes the form

$$F^T \cdot_{\phi^*TS} F \in \Gamma\left(TM \otimes T^*M\right),$$

where the relevant types are

$$\begin{aligned}
g_M^{-1} &\in \Gamma\left(TM \otimes TM\right),\\
\phi^* g_S &\in \Gamma\left(\phi^*T^*S \otimes \phi^*T^*S\right),\\
F := \overline{\nabla \phi} &\in \Gamma\left(\phi^*TS \otimes T^*M\right),\\
F^T := g_M^{-1} \cdot_{T^*M} F^{(1\,2)} \cdot_{\phi^*T^*S} \phi^* g_S &\in \Gamma\left(TM \otimes \phi^*T^*S\right).
\end{aligned}$$

Here, the $(1\,2)$ superscript indicates a permutation of the tensor factors, as described in Section 16 and Section 18.

In order to fit more naturally into the strongly typed tensor formalism and facilitate calculations in a more natural way, as alluded to and developed in Section 12.1, the right Cauchy-Green tensor can be defined as a quadratic form

$$C\colon TS \otimes T^*M \to TM \otimes T^*M$$

in the following way. Let $\pi := \pi^{TS\otimes T^*M}_{S\times M}\colon TS\otimes T^*M \to S\times M$ be the bundle projection, and let

$$r \in \Gamma\left(\pi^*\left(TS \otimes T^*M\right)\right)$$

be the radial vector field (see Lemma 12.3). Let

$$\begin{aligned} p_S &:= \mathrm{Pr}^{S\times M}_S \circ \pi\colon TS \otimes T^*M \to S \text{ and} \\ p_M &:= \mathrm{Pr}^{S\times M}_M \circ \pi\colon TS \otimes T^*M \to M. \end{aligned}$$

Because $\pi = p_S \times_{TS\otimes T^*M} p_M$, it follows that $\pi^*\left(TS \otimes T^*M\right) \cong p_S^* TS \otimes p_M^* T^*M$, so that the radial vector field can be understood to have the form $r \in \Gamma\left(p_S^* TS \otimes p_M^* T^*M\right)$. Finally, define

$$C := p_M^* g_M^{-1} \cdot_{p_M^* T^*M} r^{(1\,2)} \cdot_{p_S^* T^*S} p_S^* g_S \cdot_{p_S^* TS} r.$$

This expression looks slightly horrible, but computing its various PCDs is straightforward owing to naturality properties such as Lemma 12.8.

The function C is evaluated at $\nabla\phi$ by pulling back by $\nabla\phi$, which in this case is precomposition; $(\nabla\phi)^* C = C \circ \nabla\phi$. Note that $p_S \circ \nabla\phi = \phi$ and $p_M \circ \nabla\phi = \mathrm{Id}_M$. Then by the pullback property proved in Lemma 12.3, $(\nabla\phi)^* r = \overline{\nabla\phi}$, and it follows that

$$
\begin{aligned}
& C \circ \nabla \phi \\
=\ & (\nabla \phi)^* C && \text{(definition of pullback)} \\
=\ & (\nabla \phi)^* \left(p_M^* g_M^{-1} \cdot_{p_M^* T^* M} r^{(1\,2)} \cdot_{p_S^* T^* S} p_S^* g_S \cdot_{p_S^* TS} r \right) && \text{(definition of } C\text{)} \\
=\ & (\nabla \phi)^* p_M^* g_M^{-1} \cdot_{(\nabla \phi)^* p_M^* T^* M} (\nabla \phi)^* r^{(1\,2)} && \text{(distributivity of pullback)} \\
& \cdot_{(\nabla \phi)^* p_S^* T^* S} (\nabla \phi)^* p_S^* g_S \cdot_{(\nabla \phi)^* p_S^* TS} (\nabla \phi)^* r \\
=\ & (p_M \circ \nabla \phi)^* g_M^{-1} \cdot_{(p_M \circ \nabla \phi)^* T^* M} ((\nabla \phi)^* r)^{(1\,2)} && \text{(contravariance of pullback and} \\
& \cdot_{(p_S \circ \nabla \phi)^* T^* S} (p_S \circ \nabla \phi)^* g_S \cdot_{(p_S \circ \nabla \phi)^* TS} (\nabla \phi)^* r && \text{comm. of pullback w/ } (1\,2)) \\
=\ & \mathrm{Id}_M^* g_M^{-1} \cdot_{\mathrm{Id}_M^* T^* M} \overline{\nabla \phi}^{(1\,2)} \cdot_{\phi^* T^* S} \phi^* g_S \cdot_{\phi^* TS} \overline{\nabla \phi} && \text{(pullback property of } r\text{)} \\
=\ & g_M^{-1} \cdot_{T^* M} \overline{\nabla \phi}^{(1\,2)} \cdot_{\phi^* T^* S} \phi^* g_S \cdot_{\phi^* TS} \overline{\nabla \phi} && (\mathrm{Id}_M^* TM \cong TM) \\
=\ & F^T \cdot_{\phi^* TS} F,
\end{aligned}
$$

which is the right Cauchy-Green deformation tensor defined previously.

3.3.4 Hyperelastic Material Stored Energy Function

The stored energy function for a hyperelastic material is a function of the pointwise values of the deformation gradient $\nabla \phi$. Note that if $m \in M$, then

$$\nabla \phi (m) \in T_{\phi(m)} S \otimes T_m^* M,$$

which is the tensor identification of a linear map of the form

$$T_m M \to T_{\phi(m)} S.$$

Because $\phi (m)$ could be an arbitrary element of S, and the value of $\nabla \phi$ in a neighborhood of m can be picked so that $\nabla \phi (m)$ is an arbitrary element of $T_{\phi(m)} S \otimes T_m^* M$, it follows that the tensor bundle

$$TS \otimes T^* M \to S \times M$$

quantifies all possible local deformations (up to first order) of maps of the form $M \to S$ without requiring knowledge of what particular map it is. Thus $TS \otimes T^* M$ is the appropriate domain for the stored energy function;

$$\overline{U} \colon TS \otimes T^* M \to \mathbb{R}.$$

Under the assumption of material frame indifference (see Section 2.3), it follows that there is a unique function

$$U\colon TM \otimes T^*M \to \mathbb{R}$$

such that

$$\overline{U}\,(A) = U\left(A^T \cdot_{TS} A\right),$$

for all $A \in TS \otimes T^*M$, where $A^T := g_M^{-1} \cdot A^{(1\,2)} \cdot g_S$. This means that

$$\overline{U} \circ F = U \circ C \circ F,$$

where F and C are the deformation gradient and the right Cauchy-Green deformation tensor respectively.

3.3.5 Homogeneous Hyperelastic Materials

Recall from Section 1.6 that in standard hyperelastic theory, a hyperelastic material with stored energy function $W\,(C, X)$ is homogeneous if $\frac{\partial W}{\partial X} = 0$, where X denotes the material point parameter. Conceptually, this is understood as "W is constant along variations of the material point", and models a material whose behavior doesn't explicitly depend on the material point.

In the strongly typed formalism developed in this section, the material point and the right Cauchy-Green tensor are not decoupled; the right Cauchy-Green tensor field C takes values in $TM \otimes T^*M$, in which the material point is encoded. For brevity, let $B := TM \otimes T^*M$. The material point is recovered from C via the bundle projection

$$\pi_M^B\colon B \to M.$$

The general lack of a global trivialization for the bundle B creates a subtlety in generalizing the homogeneity condition. What is necessary is to specify what "along variations of the material point" means within B.

The solution is to create a splitting of TB into "horizontal" and "vertical" subbundles, derived from the Levi-Civita connection ∇^{TM}. For brevity, let $\pi := \pi_M^B$. Define

$$h := \overline{\nabla}\,\pi \in \Gamma\left(\pi^* TM \otimes T^* B\right).$$

This should be thought of as the horizontal projection[11]. The "vertical" variations are variations of elements of B along the fibers of B, which correspond to variations in C without variations in the material point. This is the vertical subbundle

$$VB := \ker h \leq TB,$$

which is a structure that does not require anything beyond the bundle projection π itself. Let ∇^B be the naturally induced linear covariant derivative on $B \equiv TM \otimes T^*M$. To define the vertical projection, the covariant derivative is necessary. Define

$$v \in \Gamma\left(\pi^* B \otimes T^* B\right),$$
$$v \cdot \delta_\epsilon b := \nabla^{(\pi \circ b)^* B}_{\delta_\epsilon} \overline{b},$$

where $\epsilon \mapsto b\left(\epsilon\right) \in B$ gives a tangent vector (i.e. variation) $\delta_\epsilon b \in T_{b(0)} B$. Define

$$H := \ker v \leq TB.$$

Then $TB = H \oplus VB$, and h and v, thought of as endomorphisms of TB, when restricted to H and V respectively are isomorphisms $H \to \pi^* TM$ and $VB \to \pi^* E$. This gives a bundle splitting via the vector bundle isomorphism $h \oplus v$;

$$h \oplus v \colon TB = H \oplus VB \to \pi^* TM \oplus \pi^* B.$$

The "horizontal" variations (elements of H) then correspond to variations along the material direction, and provides the necessary structure for defining the material homogeneity condition on the stored energy function $U \colon B \to \mathbb{R}$. The advantage of quantifying variations in the bundles $\pi^* TM$ and $\pi^* B$ instead of in H and VB is that there are no iterated tangent bundle constructions, the types are placed into the pullback formalism, and the types are more refined, providing insight into how the objects can or should be used. The use of PCDs (see Proposition 22.4) is what will quantify the change in U along variations in the material point. The total derivative $\nabla U \in \Gamma\left(T^* B\right)$ will be split using h and v to form the PCDs

$$U_{,h} \in \Gamma\left(\pi^* T^* M\right) \text{ and } U_{,v} \in \Gamma\left(\pi^* B^*\right)$$

[11] Technically speaking, the tensor field h represents a surjective vector bundle morphism $TB \to \pi^* TM$, but is not an endomorphism of TB, and therefore can't be considered a projection in the standard sense of a projection P satisfying $P \circ P = P$ and $P\mid_{\text{image}(P)} = \text{Id}_{\text{image}(p)}$.

defined implicitly[12] by

$$\nabla U = U_{,h} \cdot h + U_{,v} \cdot v.$$

Now that the PCDs of U are defined, the homogeneity condition is stated easily; the hyperelastic material with stored energy function $U\colon B \to \mathbb{R}$ is **homogeneous** if

$$U_{,h} = 0.$$

In the standard theory of hyperelasticity, M is a vector space, and its tangent bundle TM is naturally globally trivializable as $TM \cong M \rtimes M$ (this is a trivial bundle with fiber M and base space M; see Section (13)). Then

$$B = TM \otimes T^*M \cong (M \rtimes M) \otimes (M^* \rtimes M) \cong (M \otimes M^*) \rtimes M,$$

This makes

$$\begin{aligned} TB &\cong T\,(M \otimes M^*) \oplus TM \\ &\cong [(M \otimes M^*) \rtimes (M \otimes M^*)] \oplus [M \rtimes M] \\ &\cong [(M \otimes M^*) \oplus M] \rtimes B, \end{aligned}$$

which has a natural vertical and horizontal projection (to $(M \otimes M^*) \rtimes B$ and $M \rtimes B$ respectively). The resulting horizontal PCD has the form

$$U_{,h} \in \Gamma\,(M^* \rtimes B)\,,$$

which is really a function $U_{,h}\colon B \to M^*$, which is the same object as $\frac{\partial U}{\partial X}$, if X denotes the parameter for the M factor of $B \cong (M \otimes M^*) \rtimes M$. Thus the PCD-based formulation reduces to the standard formulation when the standard theory applies.

3.3.6 Isotropic Hyperelastic Materials

Recall from Section 1.6 that in standard hyperelastic theory, a hyperelastic material with stored energy function $W\,(C, X)$ is **isotropic** if $W\left(R^T C R, X\right) = W\,(C, X)\,,$ for all $X \in M$, symmetric $C\colon T_X M \to T_X M$ and orientation-preserving rotations

[12] The PCDs $U_{,h}$ and $U_{,v}$ can also be defined explicitly by $U_{,h} \oplus U_{,v} = \nabla U \cdot (h \oplus v)^{-1}$, since $h \oplus v$ is a vector bundle isomorphism.

$R\colon T_X M \to T_X M$. Recall that this invariance property implies that W depends on C only through the tensor invariants $\sigma_1(C), \ldots, \sigma_n(C)$, where $n = \dim M$. This result is entirely linear algebraic.

In the Riemannian setting, the invariance property just mentioned applies directly, because it is just applying the linear algebraic result to a single fiber at a time. Thus there is a unique function $\overline{U}\colon \mathbb{R}^n \rtimes M \to \mathbb{R}$ such that

$$\overline{U}\left(\sigma_1(b), \ldots, \sigma_n(b), \pi_M^B(b)\right) = U(b)$$

for all $b \in B$. As in the standard theory, an **incompressible material** is one having no local change in volume, which is expressed via tensor invariants as requiring that $\sigma_n(C) = 1$ (i.e. $\det(C) = 1$). In this case, $\overline{U}$ does not depend on $\sigma_n(b)$.

If the hyperelastic material is also homogeneous, then there is no explicit dependence on the M factor, so $\overline{U}$ is only a function of tensor invariants.

3.4 Solutions to Particular Problems in Hyperelastic Mechanics

This exposition will concern itself only with solutions to problems regarding the **hyperelastic string** model, whose defining property is simply that $\dim M = 1$ (though the dimension of the spatial manifold S remains unrestricted). Assuming M is connected and compact, M must be diffeomorphic to a real interval or a circle. Because hyperelasticity is defined in terms of stored energy functions, problems may be posed in the Lagrangian formalism (i.e. calculus of variations), which allows some particularly powerful tools to be applied.

In order to solve particular problems involving hyperelasticity, a particular hyperelastic material must be defined via its stored energy function. To this end, a set of criteria (see Section Section 7) on a physically reasonable material will be given, and from this set of criteria, particular mathematical properties derived. Various hyperelastic material models, most notably the Mooney-Rivlin material (a model for a 3-dimensional solid), will be discussed (see Section 9.2) and generalized to the arbitrary-dimensional setting (see Section 9.3). A particular hyperelastic string material will be modeled by deriving its stored energy function from the stored energy function for a material having dimension equal to the spatial manifold (see Section Section 9).

The particular problems posed and solved in this exposition are as follows.

- Embedding into an arbitrary spatial manifold a string whose energy functional only includes the stored energy term (i.e. no potential energy arising from e.g. gravity, and no kinetic energy), and obtaining a static solution (i.e. equilibrium). Free-endpoint and the fixed-endpoint boundary conditions are each addressed. See Section Section 8.

- Embedding into the 2-dimensional graph manifold S defined by the paraboloid $y = f(X) := -|X|^2$, where $X = (X^0, X^1)$ are graph coordinates, a radially symmetric, closed-loop string problem with potential energy function derived from a gravitational potential on S will be posed. In the static problem, the Euler-Lagrange equation is an ODE which reduces to an algebraic equation due to the radial symmetry of the string (see Section 10.2). In the dynamic problem, the Euler-Lagrange equation is a PDE that reduces to an ODE in the time-parameterized radius of the string, and a conserved quantity is derived and used to express solutions as phase portraits (see Section Section 11).

- With the same graph manifold and potential energy as the previous paraboloid problem, a static, fixed-endpoint string problem will be posed and an approximate solution obtained numerically using the variational structure present in the problem – minimizing the energy functional. An iterative algorithm for obtaining the numerical approximation will be presented, and two families of approximate solutions (varying the dimension of the numerical approximation and varying the parameter of gravitational potential) will be shown (see Section 10.3).

4 Conclusion and Future Work

This exposition developed the necessary theory to pose problems in hyperelasticity using the tools of strongly typed tensor calculus and a theory of Riemannian calculus of variations. The following are areas for further research and development, or areas in which the author already has preliminary results.

- Solutions to problems in solid hyperelasticity – in particular, "what happens when you push a cubic meter of jello into a wormhole?" – while only the energy functionals and Euler-Lagrange equations for hyperelastic strings were fully formulated, the corresponding derivations for solid hyperelasticity (i.e. where the material and spatial manifolds have the same dimension) follow analogously. The algorithm for numerical approximation of solutions to static boundary condition problems as stated in Section 10.3.3 can be applied in the solid case.

- Coordinate-free formulas for the tensor invariants and their partial covariant derivatives – isotropic hyperelastic materials have stored energy functions that depend only on the tensor invariants of the right Cauchy-Green deformation tensor field and the material point. Calculation of the first and second variations of the energy functional for a problem involving such a material requires computation of the derivatives of these tensor invariants. Using an natural tensorial formulation of the tensor invariants (i.e. not involving a choice basis or coordinates), the author has applied the strongly typed tensor formalism to compute the horizontal and vertical partial covariant derivates of said tensor invariants. In particular, the horizontal derivative of each invariant is zero. Use of these formulas is necessary to pose the first and second variations and Euler-Lagrange equation within the coordinate-free style that the strongly typed tensor formalism uses. The formulas are also useful when formulating arbitrary-dimensional hyperelasticity.

- Implementation of the physics portion of visualization of curved spaces – the original motivating application is computer visualization of curved spaces from within. The author has already implemented code for the visualization component. The theory developed within this exposition will now allow a design to be made for implementing the physical component, thereby enriching the interactive environment.

- Design for a real-time interactive 2-dimensional video game in curved space – with few notable exceptions, all video game design is done using a Euclidean vector space as the model for the game world. Topological features not found in flat space, such as the following.

  - Wormholes (i.e. topological handles), which could be orientation preserving or reversing.
  - Covering spaces, quotient spaces, and other spaces that defy human spatial intuition.

- Spaces having positive curvature bounded below or negative curvature bounded above, giving geodesics that all converge or all diverge respectively.
- Dynamically-changing spatial curvature, so that spatial features can change in real time.
- Dynamically-changing spatial topology, such as the appearance/disappearance of wormholes.

Part II

Solutions to Particular Problems in Hyperelastic Mechanics

5 Notation/Conventions

This section will outline the notation and conventions used for the rest of Part II.

Let (M, g_M) and (S, g_S) be the material and spatial Riemannian manifolds, where $\dim M = 1$, and let I be a compact interval of $\mathbb{R}$ used to parameterize time. Let m be a coordinate on M such that $g_M = dm \otimes dm$, and let t be the standard coordinate on I. The "domain" manifold is then $N := M \times I$, which parameterizes the material point and the time, and $E := TS \otimes T^*N$ is the bundle in which the deformation gradients reside. Let $\pi \colon E \to S \times N$ be its bundle projection. Let $p_S := \mathrm{Pr}_S^{S\times N} \circ \pi \colon E \to S$ and $p_N := \mathrm{Pr}_N^{S\times N} \circ \pi \colon E \to N$ be "utility" maps, useful in later constructions and calculations. Let $r \in \Gamma(\pi^* E)$ be the canonical radial vector field on E.

Let $\phi \colon N \to S$ be the string embedding. There is a useful type refinement $\overline{\phi} = \phi \times_N \mathrm{Id}_N$; with the total derivative of ϕ being $\nabla \phi \in \Gamma_{\overline{\phi}}(E)$, having the type refinement $\overline{\nabla \phi} \in \Gamma\left(\overline{\phi}^* E\right) \cong \Gamma(\phi^* TS \otimes T^* N)$. Define $\phi_{,m} := \overline{\nabla \phi} \cdot_{T^*N} \partial_m \in \Gamma(\phi^* TS)$ and define $\phi_{,t} \in \Gamma(\phi^* TS)$ analogously. Let $|\phi_{,m}|_{g_S}$ define the **[local] stretch**. The pointwise kinetic energy of the string is $B(\nabla \phi) := |\phi_{,t}|^2$ (assuming unit density) and the squared stretch of the string is $C(\nabla \phi) := |\phi_{,m}|^2$; this is the 1-dimensional equivalent of the right Cauchy-Green deformation tensor. These can be written nicely in terms of the metrics and radial vector field; $B = r \cdot_{\pi^* E^*} \pi^* (g_S \boxtimes (\partial_t \otimes \partial_t)) \cdot_{\pi^* E} r$ and $C = r \cdot_{\pi^* E^*} \pi^* (g_S \boxtimes (\partial_m \otimes \partial_m)) \cdot_{\pi^* E} r$. While these formulations of B and C seem unwieldy, they turn out to give very natural computations for their various derivatives which are used in the Euler-Lagrange equation.

Let $U \colon \mathbb{R}^+ \to \mathbb{R}$ be the function which gives the stored energy in terms of the squared stretch. The stored energy function is then $W := U \circ C \equiv C^* U \colon E \to \mathbb{R}$. Let $Q \colon S \to \mathbb{R}$ be a potential energy density function on the spatial manifold, let $q(x) := x^{-1/2}$ (used for converting potential energy density into potential energy). This induces a potential energy function $P := (q \circ C)(Q \circ p_S) : E \to \mathbb{R}$.

6 Lagrangian, Energy Functional, and Euler-Lagrange Equation

In the general setting, with Lagrangian $L\colon E \to \mathbb{R}$ and energy functional

$$\begin{aligned} \mathcal{L}\colon C^{\infty}(N, S) &\to \mathbb{R}, \\ \phi &\mapsto \int_N L \circ \overset{\nabla}{\nabla}\phi \, dV_N, \end{aligned}$$

the derived Euler-Lagrange equation has the form

$$(\overset{\nabla}{\nabla}\phi)^* L_{,\sigma} - \operatorname{Div}_N\left((\overset{\nabla}{\nabla}\phi)^* L_{,v}\right) = 0 \text{ on } N.$$

In the full hyperelastic string problem, the Lagrangian is constructed in the usual manner for a mechanical system via the various energy terms;

$$L := K - (W + P),$$

and by calculations done in Section 12.3.2, the Euler-Lagrange equation is

$$\begin{aligned} 0 = &- |\phi_{,m}|^{-1} \phi^* \operatorname{Grad} Q \\ &- |\phi_{,m}|^{-3} \left(\phi^* \nabla Q \cdot_{\phi^* TS} \phi_{,m}\right) \phi_{,m} \\ &+ \left(3\phi^* Q \, |\phi_{,m}|^{-5} + 4U''\left(|\phi_{,m}|^2\right)\right) \left(\phi_{,m} \cdot_{\phi^* T^* S} \phi^* g_S \cdot_{\phi^* TS} \nabla_{\partial_m} \phi_{,m}\right) \phi_{,m} \\ &- \left(\phi^* Q \, |\phi_{,m}|^{-3} - 2U'\left(|\phi_{,m}|^2\right)\right) \nabla_{\partial_m} \phi_{,m} + \nabla_{\partial_t} \phi_{,t}. \end{aligned}$$

Particular boundary conditions will be posed and dealt with later.

7 Criteria for a Physically Reasonable Material

Based on common-sense physical intuition, the following assumptions will be made about a material[13] whose behavior is defined by the stored energy function U. Recall that U is a function of the squared stretch of the string. A material will be called **physically reasonable** if it satisfies the following properties.

1. The material cannot be infinitely compressed to attain zero length.

[13] The criteria detailed here are applicable to nonhomogeneous materials, even though only the homogeneous case is addressed in this exposition.

2. The material has a unique **rest stretch** equal to one. Note that the [local] stretch is defined with respect to the metrics on both the material and spatial manifolds, thus the "shape" of the string material is dictated by the material manifold's metric.

3. Nonzero forces are applied at each point when the stretch deviates from the rest stretch, and the forces cause the material to tend to move toward the rest stretch.

These assumptions, understood to be local conditions, can be translated into mathematical statements about $U(\lambda)$ and $\phi(m)$.

- Property 1 requires that $\lim_{\lambda \to 0^+} U(\lambda) = +\infty$, or put in words, the stored energy must approach infinity as the squared stretch $\lambda = |\phi'|^2$ approaches zero (approaches infinite compression). This implies that ϕ is regular (i.e. ϕ' is nonvanishing).

- Property 2 is therefore expressed as the function $\lambda \mapsto U(\lambda)$ having a unique critical point at $\lambda = 1$. With respect to the derivative, $U'(\lambda) = 0$ if and only if $\lambda = 1$.

- Finally, Property 3 will cause the material to apply forces to decrease stretching when above the rest stretch, and apply forces to increase stretching (decrease compression) when below the rest stretch. This is expressed mathematically as saying that the minimum stored energy occurs at the rest stretch, and that the stored energy increases monotonically as the stretch deviates further from the rest stretch. Put concisely, $U'(\lambda) > 0$ for $\lambda > 1$ and $U'(\lambda) < 0$ for $\lambda < 1$. While it is certainly true that a convex function $\lambda \mapsto U(\lambda)$ with $U'(\lambda) = 0$ satisfies this property, the converse is not true.

8 Full Static Solution for Zero Potential Energy

For this section, it will be assumed that the potential energy function Q is zero and that all time derivatives are zero. Except for the following subsection, which will present a full solution to the free endpoint problem, it will be assumed that the endpoints of the string are fixed.

Here, because there is no time dependence, it can be assumed that ϕ has the form $\phi \colon M \to S$, and that the univariate derivative $\phi' := \overline{\nabla \phi} \cdot \partial_m \in \Gamma(\phi^* TS)$ is well-defined. Under the assumptions given, the general Euler-Lagrange equation (calculated in Section 12.3.2) reduces to

$$\begin{aligned}0 &= \mathrm{Div}_M \left((\phi')^* L_{,v} \right) \\ &= \nabla_{\partial_m} \left((\phi')^* L_{,v} \right) \cdot \partial_m,\end{aligned}$$

and because this is an ordinary derivative, this means that $(\phi')^* L_{,v}$ is constant on M.

8.1 Free Endpoint Solution

If the endpoints of the string are free, there is an additional Euler-Lagrange equation for the boundary of the string.

$$(\phi')^* L_{,v} = 0 \text{ on } \partial M,$$

and because $(\phi')^* L_{,v}$ is constant on M, this implies that $(\phi')^* L_{,v} = 0$ on M. In particular, this takes the form

$$0 = -2U' \left(|\phi'|^2 \right) \phi' \cdot_{\phi^* T^* S} \phi^* g_S \otimes \partial_m,$$

and via contraction with the invertible tensor field $-\frac{1}{2}\phi^* g_S^{-1} \otimes dm$, simplifies to

$$0 = U' \left(|\phi'|^2 \right) \phi'.$$

Because $|\phi'|$ is nonvanishing, it follows that $U' \left(|\phi'|^2 \right)$ is identically zero. By the physically reasonable material criteria, this implies that $|\phi'|^2 = 1$. This condition makes the Euler-Lagrange equation otherwise useless because it gives no directional information. Thus any unit-stretch curve ϕ is a solution. Call this the **slack string** solution. There is no tension or compression. This non-uniqueness of solution can be reconciled with the physical model in that a string provides no resistance to bending, and therefore should not prefer any particular bent configuration over another.

8.2 Fixed Endpoint Solution

8.2.1 A Conserved Quantity

For the rest of this section assume that the endpoints of the string are fixed. A consequence of this is that there is no Euler-Lagrange equation for the boundary of M. Recall that the main Euler-Lagrange equation is[14] simply

$$\nabla_{\partial_m} \left(2U' \left(|\phi'|^2 \right) \phi' \right) = 0.$$

[14] after being contracted with the invertible tensor field $\phi^* g_S^{-1} \otimes dm$ on the right side.

This looks like one factor of a product rule, from which it is not difficult to determine what a conserved quantity looks like. Note that $\nabla^{TS} g_S = 0$ due to the metric compatibility of ∇^{TS}, so $\nabla \phi^* g_S = \phi^* \nabla g_S \cdot \overline{\nabla \phi} = 0$, meaning that tensor contraction with $\phi^* g_S$ can commute with covariant differentiation freely. Furthermore, g_S is symmetric, so $\phi^* g_S$ is also, allowing the two terms of the product rule to combine.

$$\begin{aligned}
\partial_m \left| U'\left(|\phi'|^2\right) \phi' \right|^2 &= 2 \left| U'\left(|\phi'|^2\right) \phi' \right| \partial_m \left[\left(U'\left(|\phi'|^2\right) \phi' \right) \cdot \phi^* g_S \cdot \left(U'\left(|\phi'|^2\right) \phi' \right) \right] \\
&= 2 \left| U'\left(|\phi'|^2\right) \phi' \right| \nabla_{\partial_m} \left(U'\left(|\phi'|^2\right) \phi' \right) \cdot \phi^* g_S \cdot \left(U'\left(|\phi'|^2\right) \phi' \right) \\
&\quad + 2 \left| U'\left(|\phi'|^2\right) \phi' \right| \left(U'\left(|\phi'|^2\right) \phi' \right) \cdot \phi^* g_S \cdot \nabla_{\partial_m} \left(U'\left(|\phi'|^2\right) \phi' \right) \\
&= 2 \left| U'\left(|\phi'|^2\right) \phi' \right| \left(U'\left(|\phi'|^2\right) \phi' \right) \cdot \phi^* g_S \cdot \nabla_{\partial_m} \left(2U'\left(|\phi'|^2\right) \phi' \right) \\
&= 2 \left| U'\left(|\phi'|^2\right) \phi' \right| 0 \cdot \phi^* g_S \cdot \left(U'\left(|\phi'|^2\right) \phi' \right) \\
&= 0.
\end{aligned}$$

Because $\left| U'\left(|\phi'|^2\right) \phi' \right|^2$ is a scalar function with vanishing derivative, it is constant, and in particular because it is the square of something, it is non-negative, say equal to $b^2 \geq 0$.

$$\begin{aligned}
b^2 &= \left| U'\left(|\phi'|^2\right) \phi' \right|^2 \\
&= \left(U'\left(|\phi'|^2\right) \right)^2 |\phi'|^2 .
\end{aligned} \tag{8.1}$$

8.2.2 A Qualitative Deduction

There are two cases to consider:

1. If $b = 0$, then the right-hand side of (8.1) equals zero. But $|\phi'| \neq 0$ (by assumption that infinite compression is forbidden), which implies that $U'\left(|\phi'|^2\right) \equiv 0$, and therefore, as in the free endpoint solution, the string is slack, and any unit-stretch curve ϕ connecting the given endpoints is a solution. This case is therefore considered to be fully solved.

2. If $b \neq 0$, then because $|\phi'| > 0$, it follows that $U'\left(|\phi'|^2\right)$ is nonvanishing, meaning that it is either entirely positive or entirely negative. The string lies along a geodesic between the two endpoints (a fact that will be proved in the next section).

   (a) If $U'\left(|\phi'|^2\right) < 0$, this corresponds to $|\phi'| < 1$, meaning that the string is under **compression**. Intuitively, one would expect that any perturbation of this configuration out of the geodesic path will lower the energy and tend towards a slack string, making this an unstable equilibrium.

(b) If $U'\left(|\phi'|^2\right) > 0$, this corresponds to $|\phi'| > 1$, meaning that the string is under **tension**. If the geodesic which this string lies along is globally length minimizing, then intuitively it stands to reason that this is a stable equilibrium.

8.2.3 Solution Decomposition

Cases 2a and 2b (compression and tension) will be handled simultaneously here. The conserved quantity determines a first-order nonlinear ODE for the stretch of ϕ. The direction of ϕ will be provided by the fact that ϕ is a reparameterization of a geodesic between the endpoints of the string. Questions about the uniqueness (or lack thereof) of geodesics between the given endpoints fall to general differential geometric theory.

Let $\Sigma(m) := U'\left(|\phi'(m)|^2\right)$ for brevity. The Euler-Lagrange equation is

$$0 = \nabla_{\partial_m}\left(2\Sigma\,\phi'\right)$$

and renders the ODE

$$\begin{aligned} 0 &= \nabla_{\partial_m}\left(\Sigma\,\phi'\right) \\ &= \Sigma'\,\phi' + \Sigma\,\nabla_{\partial_m}\phi' \\ \implies \nabla_{\partial_m}\phi' &= -\frac{\Sigma'}{\Sigma}\phi', \end{aligned}$$

and by Lemmas Lemma 12.10 and Lemma 12.9, this implies that ϕ is the reparameterization of a geodesic, and that $-\Sigma'/\Sigma = \frac{d}{dm}\left(\log|\phi'|\right)$. The fact that ϕ is the reparameterization of a geodesic determines the image of the solution ϕ, while the equation $-\Sigma'/\Sigma = \frac{d}{dm}\left(\log|\phi'|\right)$ can be shown to imply that $U'\left(|\phi'|^2\right)|\phi'|$ is constant, a fact which gives no new information. Depending on the form of U, for example the definition used later in this exposition, this may render an algebraic equation for $|\phi'|$.

In the particular case where $U(\lambda) = \lambda - 2 + \lambda^{-1}$, it follows that $U'(\lambda) = 1 - \lambda^{-2}$, and therefore that

$$\left(1 - |\phi'|^{-4}\right)|\phi'| = c$$

for some constant $c \in \mathbb{R}$, and the solutions in $|\phi'|$ are zeros of the polynomial $|\phi'|^4 - c\,|\phi'|^3 - 1$. In particular, that $|\phi'|$ is constant with respect to its argument m and therefore that ϕ is a legitimate geodesic. This makes sense in light of the fact that the material is homogeneous and therefore the forces should be expected to distribute evenly along the string.

9 Modeling a Particular String Material

A hyperelastic string is a 1-dimensional material which is embedded into a manifold of arbitrarily high dimension n; it will be assumed that $n \geq 2$. The string can be modeled as a hyperelastic solid (n-dimensional) having infinitesimal cross section. As such, the stored energy function for a hyperelastic string can be derived from the stored energy function for the solid, modeling the string's behavior as the behavior of the solid under uniaxial stress (forces applied along a single axis). A solid under uniaxial stress is a highly symmetric configuration which can be expressed very easily using the principle stretches and principle axes of the right Cauchy-Green deformation tensor; the principle stretch in the direction of uniaxial stress is given, say $\lambda > 0$, and the other principle stretches are assumed to be equal, say $\mu > 0$. If the material is incompressible, then $1 = \det(C) = \lambda\mu^{n-1}$, so $\mu = \lambda^{-1/(n-1)}$, and the solid's stress configuration has a single parameter.

9.1 Approach to and Validity of Generalization

The theory of solid hyperelasticity is quantified in terms of the local deformations of an elastic body. This is a concept which is expressed via derivatives and naturally generalizes to the setting of Riemannian manifolds. The strongly typed pull-back formalism of tensor calculus assists in making the correct constructions.

The deformation gradient is the total derivative of the material embedding, and the right Cauchy-Green deformation tensor is polynomial in the deformation gradient. In the setting of Riemannian manifolds, where the material embedding (configuration of the body in space) has the form

$$\phi\colon M \to S,$$

the deformation gradient is

$$F := \overline{\nabla\,\phi} \in \Gamma\left(\phi^* TS \otimes T^* M\right),$$

and the right Cauchy-Green tensor [field] is

$$C := F^T \cdot F \in \Gamma\left(TM \otimes T^*M\right),$$

where $A^T := g_M^{-1} \cdot A^{(1\,2)} \cdot g_S$ for $A \in \Gamma\left(\phi^* TS \otimes T^*M\right)$. The tensor fields F and C are quantifications of the "local stretch" and "squared local stretch" of the embedding ϕ, where C necessarily takes into account the metric structure of both manifolds.

9.2 Particular Forms of Stored Energy Functions

The general approach to approximating the stored energy function for real materials was formulated by R.S. Rivlin in [Riv97, pg 328] in the $M = \mathbb{R}^3$ setting, starting with a power series about $C = \mathbb{I}$, where $\mathbb{I}$ denotes the identity matrix.

$$U(C) = \sum_{i,j,k=0}^{\infty} a_{ijk} \left(\sigma_1(C) - 3\right)^i \left(\sigma_2(C) - 3\right)^j \left(\sigma_3(C) - 1\right)^k,$$

where $a_{ijk} \in \mathbb{R}$ are the **material constants**. Note that $\sigma_1(\mathbb{I}) = 3$, $\sigma_2(\mathbb{I}) = 3$, and $\sigma_3(\mathbb{I}) = 1$. No constant term is needed because any stored energy functions which differ by a constant give identical material behavior. Therefore it can be assumed that $U(\mathbb{I}) = 0$. Here, no assumption of incompressibility is made.

In determining the coefficients of such an approximation for a real material, a finite sum out to some particular order is used.

$$U(C) = \sum_{i,j,k=0}^{N} a_{ijk} \left(\sigma_1(C) - 3\right)^i \left(\sigma_2(C) - 3\right)^j \left(\sigma_3(C) - 1\right)^k.$$

The material constants a_{ijk} are then unknown parameters to be determined by a series of stress-strain measurements in various configurations of the material.

For incompressible materials, $\sigma_3(C) = 1$, and the stored energy is

$$U(C) = \sum_{p,q=0}^{N} a_{pq} \left(\sigma_1(C) - 3\right)^p \left(\sigma_2(C) - 3\right)^q$$

(see [Riv97, pg 158]). This is called the **generalized Rivlin model**.

A special case of the stored energy function for a hyperelastic material is a linear approximation about the undeformed state. Such an approximation would only be valid for small deformations of the material (see [Riv97, pg 191]). This renders the stored energy function

$$U(C) = a_1\left(\sigma_1(C) - 3\right) + a_2\left(\sigma_2(C) - 3\right) + a_3\left(\sigma_3(C) - 1\right)$$

for compressible materials, and

$$U(C) = a_1(\sigma_1(C) - 3) + a_2(\sigma_2(C) - 3)$$

for incompressible materials. The latter is known as a **Mooney-Rivlin** material (see [Riv97, pg xxxi] and [Moo40, equation (14)])[15].

Finally, a further simplification is to eliminate the dependence on $\sigma_2(C)$ as well, rendering the stored energy function

$$U(C) = a(\sigma_1(C) - 3).$$

This is the stored energy function for a **Neo-Hookean** material (see [MH83, pg 201]).

9.3 Generalizing Particular Stored Energy Functions to Arbitrary Manifolds

Given the approaches to approximating the stored energy function for an isotropic, homogeneous, hyperelastic material discussed in Section 9.2, and the fact that the entire existing theory is based on local deformation, the generalization to arbitrary manifolds is straightforward. The stored energy can be expressed by taking a power series in the tensor invariants about the identity tensor field, which would represent an undeformed state. Recall that $\sigma_j(\mathbb{I}) = \binom{n}{j}$, and that $U\colon TM \otimes T^*M \to \mathbb{R}$. The stored energy function is

$$U(C) = \sum_{I_1,\ldots,I_n=0}^{\infty} a_{I_1\ldots I_n} \prod_{k=1}^{n} \left(\sigma_k(C) - \binom{n}{k}\right)^{I_k},$$

where $a_{I_1\ldots I_n} \in \mathbb{R}$ are the material constants.

Taking a partial sum of the power series and assuming incompressibility renders the form

$$U(C) = \sum_{I_1,\ldots,I_{n-1}=0}^{N} a_{I_1\ldots I_{n-1}} \prod_{k=1}^{n-1} \left(\sigma_k(C) - \binom{n}{k}\right)^{I_k},$$

which is analogous to the generalized Rivlin model.

Taking only the first-order terms gives a generalization of the stored energy function of the Mooney-Rivlin material, and has the form

$$U(C) = \sum_{k=0}^{n-1} a_k \left(\sigma_k(C) - \binom{n}{k}\right).$$

[15] This is called "the Mooney form" by R.S.Rivlin.

Finally, the generalization of a Neo-Hookean material is made by taking only the term containing the first tensor invariant.

$$U(C) = a(\sigma_1(C) - n).$$

An interesting related fact is that if the condition of incompressibility is removed, then the energy functional

$$(\phi\colon M \to S) \mapsto \int_M a\left(\sigma_1\left(\overline{\nabla\phi}\right) - n\right)\, dV_M$$

is equivalent to the functional

$$(\phi\colon M \to S) \mapsto \int_M \frac{1}{2}\left|\overline{\nabla\phi}\right|^2\, dV_M,$$

whose critical points are harmonic maps from M to S.

9.4 Deriving the String's Stored Energy Function

The incompressible Mooney-Rivlin solid described above will be used to derive the stored energy function for the corresponding hyperelastic string. Because the principle stretches (eigenvalues of C) are $\lambda, \mu, \dots, \mu$, and $\sigma_k(C)$ is the kth elementary symmetric polynomial in the principle stretches, it follows that if $k < n$, then

$$\begin{aligned}
\sigma_k(C) &= \sigma_k(\lambda, \mu, \dots, \mu) \\
&= \binom{n-1}{k-1}\lambda\mu^{k-1} + \binom{n-1}{k}\mu^k \\
&= \left[\binom{n-1}{k-1}\lambda + \binom{n-1}{k}\mu\right]\mu^{k-1}.
\end{aligned}$$

Because the hyperelastic string's stored energy is only a function of the uniaxial stretch, λ, it follows that the induced stored energy function is

$$U(\lambda) = \sum_{k=1}^{n-1} a_k\left(\left[\binom{n-1}{k-1}\lambda + \binom{n-1}{k}\mu\right]\mu^{k-1} - \binom{n}{k}\right). \tag{9.1}$$

9.5 A Physically Reasonable Material

It will be shown in this section that the hyperelastic string material derived from the n-dimensional incompressible Mooney-Rivlin material is physically reasonable in the sense described in Section Section 7.

$$\lim_{\lambda\to 0^+} U(\lambda) = \lim_{\lambda\to 0^+} \sum_{k=1}^{n-1} a_k \left(\left[\binom{n-1}{k-1} \lambda \mu^{k-1} + \binom{n-1}{k} \mu^k \right] - \binom{n}{k} \right)$$
$$= \lim_{\lambda\to 0^+} \sum_{k=1}^{n-1} a_k \left(\left[\binom{n-1}{k-1} \lambda^{-1-(k-1)/(n-1)} + \binom{n-1}{k} \lambda^{-k/(n-1)} \right] - \binom{n}{k} \right).$$

Because each coefficient is positive, each term is either constant or a power of λ having a pole at 0, and each pole has the desired asymptotic behavior (approaching $+\infty$ as λ approaches 0), the limit is $+\infty$, so the material satisfies Property 1.

Recall that $\mu(\lambda) = \lambda^{-1/(n-1)}$, so $\mu'(\lambda) = -\frac{1}{n-1}\mu^n$, and it follows that

$$U'(\lambda) = \sum_{k=1}^{n-1} a_k \left(\left[\binom{n-1}{k-1} + \binom{n-1}{k} \mu' \right] \mu^{k-1} \right.$$
$$\left. + \left[\binom{n-1}{k-1} \lambda + \binom{n-1}{k} \mu \right] (k-1) \mu' \mu^{k-2} \right)$$
$$= \sum_{k=1}^{n-1} a_k \left(\binom{n-1}{k-1} \mu + \binom{n-1}{k-1} \lambda (k-1) \mu' + k \binom{n-1}{k} \mu' \mu \right) \mu^{k-2}$$
$$= \sum_{k=1}^{n-1} a_k \left(\binom{n-1}{k-1} \mu - \binom{n-2}{k-2} \lambda \mu^n - \binom{n-2}{k-1} \mu^{n+1} \right) \mu^{k-2}$$
$$= \sum_{k=1}^{n-1} a_k \left(\binom{n-1}{k-1} - \binom{n-2}{k-2} - \binom{n-2}{k-1} \mu^n \right) \mu^{k-1}$$
$$= (1 - \mu(\lambda)^n) \sum_{k=1}^{n-1} a_k \binom{n-2}{k-1} \mu(\lambda)^{k-1}.$$

Because a_k, $\binom{n-2}{k-1}$, and $\mu(\lambda)^{k-1}$ are each positive for each choice of k, n, it follows that the summation is positive. Thus $\operatorname{Sgn}(U'(\lambda)) = \operatorname{Sgn}(1 - \mu(\lambda)^n)$. It is clear that $1 - \mu(\lambda)^n$, expressed in the variable λ as $1 - \lambda^{-n/(n-1)}$, is positive for $\lambda > 1$ and negative for $\lambda < 1$. Thus the material satisfies Property 3. It then follows that U has a unique critical point at $\lambda = 1$, thereby satisfying Property 2. Therefore the hyperelastic string material induced by the n-dimensional incompressible Mooney-Rivlin material is a physically reasonable material.

9.6 Examples of Derived String Energy Functions

Because the stored energy function and its derivatives show up in the Euler-Lagrange equation describing the statics and dynamics of a hyperelastic string, it will be informative to compute particular examples of the stored energy functions deriving from solids of various dimensions. The derivation (9.1) will be used for this purpose.

Deriving the hyperelastic string material from a 2-dimensional Mooney-Rivlin solid, i.e. $n = 2$, then $\mu(\lambda) = \lambda^{-1/(2-1)} = \lambda^{-1}$, and straightforward computations produce the formulas

$$\begin{aligned} U(\lambda) &= a_1\lambda - 2a_1 + a_1\lambda^{-1}, \\ U'(\lambda) &= a_1 - a_1\lambda^{-2}, \qquad \text{(stress tensor)} \\ U''(\lambda) &= 2a_1\lambda^{-3}. \end{aligned}$$

Note that because $a_1 > 0$ and $\lambda > 0$, $U''(\lambda) > 0$, so U is strictly [poly]convex (a condition stronger than that required by Property 3), and $U'(\lambda)$ has a unique zero at $\lambda = 1$. This convexity happens to be atypical, and occurs only in dimension 2.

Deriving the string material from a 3-dimensional Mooney-Rivlin solid, i.e. $n = 3$, then $\mu(\lambda) = \lambda^{-1/(3-1)} = \lambda^{-1/2}$, and straightforward computations produce the formulas

$$\begin{aligned} U(\lambda) &= a_1\lambda + 2a_2\lambda^{1/2} - 3(a_1 + a_2) + 2a_1\lambda^{-1/2} + a_2\lambda^{-1}, \\ U'(\lambda) &= a_1 + a_2\lambda^{-1/2} - a_1\lambda^{-3/2} - a_2\lambda^{-2}, \qquad \text{(stress tensor)} \\ U''(\lambda) &= -\frac{1}{2}a_2\lambda^{-3/2} + \frac{3}{2}a_1\lambda^{-5/2} + 2a_2\lambda^{-3}. \end{aligned}$$

It should be noted that in this case, U is *not* convex with respect to λ;

$$\begin{aligned} 0 = U''(\lambda) &= -\frac{1}{2}a_2\lambda^{-3/2} + \frac{3}{2}a_1\lambda^{-5/2} + 2a_2\lambda^{-3} \\ \iff a_2\lambda^{-3/2} - 3a_1\lambda^{-5/2} &= 4a_2\lambda^{-3} \\ \iff a_2^2\lambda^{-3} - 6a_1a_2\lambda^{-4} + 9a_1^2\lambda^{-5} &= 16a_2^2\lambda^{-6} \\ \iff 0 &= 16a_2^2 - 9a_1^2\lambda + 6a_1a_2\lambda^2 - a_2^2\lambda^3, \end{aligned}$$

and because a_2 takes nonnegative values, this is a nondegenerate cubic in λ, positive when $\lambda = 0$, having negative cubic coefficient, and therefore has a zero for some $\lambda > 0$. The point to be made here is that a hyperelastic string material is qualitatively different based on the dimension of the solid it was derived from.

Deriving the string material from a 4-dimensional Mooney-Rivlin solid, i.e. $n = 4$, $\mu(\lambda) = \lambda^{-1/(4-1)} = \lambda^{-1/3}$, and straightforward computations produce the formulas

$$\begin{aligned}
U(\lambda) &= a_1\lambda + 3a_2\lambda^{2/3} + 3a_3\lambda^{1/3} - 4a_1 - 6a_2 - 4a_3 + 3a_1\lambda^{-1/3} + 3a_2\lambda^{-2/3} + a_3\lambda^{-1}, \\
U'(\lambda) &= a_1 + 2a_2\lambda^{-1/3} + a_3\lambda^{-2/3} - a_1\lambda^{-4/3} - 2a_2\lambda^{-5/3} - a_3\lambda^{-2}, \\
U''(\lambda) &= -\frac{2}{3}a_2\lambda^{-4/3} - \frac{2}{3}a_3\lambda^{-5/3} + \frac{4}{3}a_1\lambda^{-7/3} + \frac{10}{3}a_2\lambda^{-8/3} + 2a_3\lambda^{-3}.
\end{aligned}$$

10 Static Problem for Nonzero Potential Energy

Now the case where the potential energy function $Q\colon S \to \mathbb{R}$ can take nonzero values will be considered. Again using the derivation of the Euler-Lagrange equation in Section 12.3.2, it follows that

$$\begin{aligned}
0 = &-\left|\phi'\right|^{-1} \phi^* \operatorname{Grad} Q \\
&-\left|\phi'\right|^{-3} \left(\phi^*\nabla Q \cdot_{\phi^*TS} \phi'\right) \phi' \\
&+\left(3\phi^*Q \left|\phi'\right|^{-5} + 4U''\left(\left|\phi'\right|^2\right)\right) \left(\phi' \cdot_{\phi^*T^*S} \phi^* g_S \cdot_{\phi^*TS} \nabla_{\partial_m}\phi'\right) \phi' \\
&-\left(\phi^*Q \left|\phi'\right|^{-3} - 2U'\left(\left|\phi'\right|^2\right)\right) \nabla_{\partial_m}\phi'.
\end{aligned}$$

10.1 A Particular Material, Manifold, and Potential Energy Density

Now a particular choice for the hyperelastic string energy function U and spatial manifold S will be made so that specific equations can be derived.

Let $f(X) := -|X|^2$ define the graph manifold S, where $X = \left(X^0, X^1\right)$ are the standard coordinates on $\mathbb{R}^2$, and where the metric g_S is induced by the embedding $X \mapsto (X, f(X))$ in $\mathbb{R}^3$. Define $Q\colon S \to \mathbb{R}$ by $Q := G\,f$, where $G \in \mathbb{R}$ is a gravitational constant. Let U be the hyperelastic string energy function derived from a 2-dimensional homogeneous Mooney-Rivlin solid (corresponding to the dimension of S); $U(\lambda) = a\left(\lambda - 2 + \lambda^{-1}\right)$. Here, $a > 0$ is the material constant.

In this case, the Lagrangian is $L := W + P$, whose first and second terms depend linearly on a and G respectively. Because dividing through by a constant does not change the Euler-Lagrange equation that is produced, it suffices to divide through by $2a$ and to define $J := G/2a$, thereby changing the stored energy function and potential energy density functions to

$$U(\lambda) = \lambda - 2 + \lambda^{-1},$$
$$Q = 2J\,f.$$

The stored energy function has derivatives

$$U'(\lambda) = 1 - \lambda^{-2},$$
$$U''(\lambda) = 2\lambda^{-3}.$$

10.2 A Radially Symmetric Solution

The chosen graph manifold and potential energy function have radial symmetry, so it makes sense to consider a radially symmetric solution.

Let $\phi(m) = \rho(\cos m, \sin m)$ in graph coordinates, where $\rho > 0$ (the case $\rho = 0$ is ruled out because infinite compression is disallowed by the criteria of a hyperelastic string material being physically reasonable). Following the calculations made in Section 12.3.6, the Euler-Lagrange [differential] equation reduces to the [non-differential] equation

$$0 = \left[\frac{4J}{1+4\rho^2} - 2\left(J\rho^{-1} + 1 - \rho^{-4}\right)\left(\frac{\rho}{1+4\rho^2}\right)\right](\cos m, \sin m).$$

Because $(\cos m, \sin m)$ is a nonvanishing vector quantity, it follows that the scalar factor that precedes it is zero. Combining it into a single fraction gives

$$\begin{aligned} 0 &= \frac{4J - 2\left(J\rho^{-1} + 1 - \rho^{-4}\right)\rho}{1+4\rho^2} \\ &= \frac{2J - 2\rho + 2\rho^{-3}}{1+4\rho^2} \\ &= -2\rho^{-3}\frac{-J\rho^3 + \rho^4 - 1}{1+4\rho^2}. \end{aligned}$$

and dividing through by $-\frac{2\rho^{-3}}{1+4\rho^2}$ gives

$$0 = \rho^4 - J\rho^3 - 1.$$

Because $\rho > 0$, this equation is equivalent to

$$J = \frac{\rho^4 - 1}{\rho^3},$$

and because the right-hand side is a continuous, unbounded function for $\rho > 0$, it follows that for any $J \in \mathbb{R}$, there is some $\rho > 0$ satisfying the equation. In other words, the polynomial $\rho \mapsto \rho^4 - J\rho^3 - 1$ has a positive, real root. Furthermore, because J is uniquely defined as a function of ρ, it follows that this positive, real root is unique, and therefore ρ (the unique, positive, real root) can be considered a function of J.

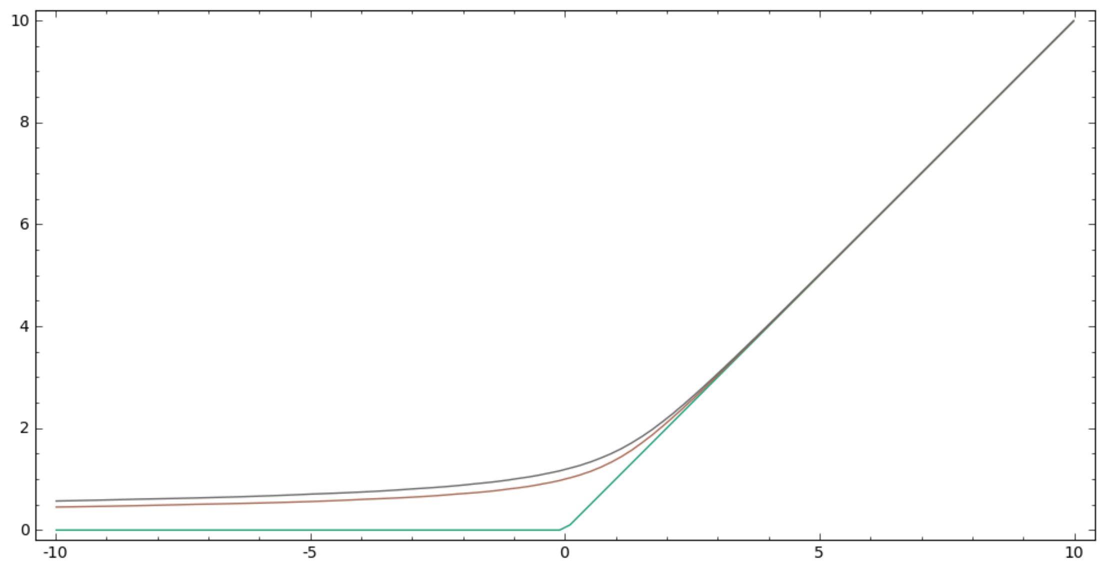


Figure 10.1: The horizontal and vertical axes of this graph denote J and ρ respectively. As a function of J, $\rho(J)$ has asymptotes $\rho = 0$ (as $J \to -\infty$) and $\rho = J$ (as $J \to \infty$), and is shown in red. The -1, 0, and 1 level set contours of the function $p(J, \rho) := \rho^4 - J\rho^3 - 1$ are green, red, and black respectively.

As $G \to \infty$, $(J, \rho) \to (\infty, \infty)$, showing that the radius increases as gravity increases. As $a \to \infty$, $(J, \rho) \to (0, \rho)$, showing that the string tends toward its rest stretch as its material constant increases. This is physically reasonable behavior.

As $G \to -\infty$, corresponding to gravity acting upward, there are still solutions, the radii of which tend asymptotically toward zero, and these correspond to unstable equilibria, where the string is compressed in a perfect circle.

10.3 Radially Asymmetric Boundary Condition Problem

Using the same graph manifold as before, a nontrivial problem is finding a solution when the endpoints of the string are fixed to prescribed points. In this case, the points chosen will be $(-0.5, 0)$ and $(0.5, 0)$ in graph coordinates X. It will be assumed that gravity is downward ($G > 0$) and that the string drapes down below the x-axis of the graph coordinates.[16]

Because the problem is variationally formulated – finding a minimizer of the energy functional $\mathcal{L}$ – it is conceptually straightforward to implement a numerical method for finding approximate solutions. Because the space of all admissible curves (i.e. having the required boundary condition) is an infinite dimensional manifold and a computer has a finite amount of memory and processing capacity, this requires picking a finite-dimensional submanifold of **admissible curves** on which to perform the numerical computation. Additionally, the problem can be parameterized by the dimension of the submanifold of admissible curves, and said submanifolds can be picked such that their union is dense in the full, infinite-dimensional manifold.

10.3.1 Form of the Admissible Curves

The current work uses a piecewise linear family of admissible curves, where the dimension of said family is related to the number of pieces. This gives piecewise differentiability, which is enough for the energy functional $\mathcal{L}$ to be well-defined. Another choice considered is piecewise cubic polynomial, which would be C^1, and look much smoother, at the cost of doubling the number of dimensions per piece; however, this is not tried in this work. In the work thus far, piecewise linear has shown to be tenable, and possibly computationally less expensive than piecewise cubic. In the ultimate application of this theory, computer visualization of manifolds from within, visual smoothness is required, so a C^1 family will be used in that case. In the following, a piecewise linear family of admissible curves will suffice.

10.3.2 Computation of the Energy Functional

The computations to find numerical approximations of solutions were done using the computer algebra system Sage[17]. Perhaps the thing most dependent on the use of symbolic calculus is the computation of the energy functional. With Sage's symbolic calculus functionality, the piecewise linear family of admissible curves was parameterized

[16] There is a trivial, unstable solution where the string drapes over the top of the paraboloid, which goes exactly along the x-axis

[17] Sage (www.sagemath.org) is a mathematical extension of the Python programming language which is capable of many operations in elementary calculus.

by formal symbolic parameters, so that relevant derivatives could be computed automatically. A complicating feature of the formulation of the energy functional is that there was no obvious indefinite integral for the Lagrangian, and symbolic integrators failed to provide one. However, a Riemann sum (or higher order integration technique) provided a symbolic approximation of the energy functional restricted to the given family of admissible curves. The specifics of this are as follows.

- Let $(a_1, \ldots, a_N)$ parameterize the family of curves (pairs of a_i values giving the (x, y) coordinates of each vertex in the piecewise linear curve).
- The curve's parameter is $m \in M$ (in this case, $M = [-1, 1]$), and the energy functional is a definite integral over M.
- Use a Riemann sum (or other technique) to divide the integrating domain into k equal-sized segments. Evaluation of the Lagrangian at the center point of each segment produces a symbolic expression in the curve parameters $(a_1, \ldots, a_N)$. The sum of such expressions is again a symbolic expression in the parameters of the family of admissible curves. This expression is a symbolic approximation of the energy functional restricted to the given family of curves. Denote this by $\mathcal{L}(a_1, \ldots, a_N)$.

10.3.3 Numerical Optimization Algorithm

Because $\mathcal{L}(a_1, \ldots, a_N)$ is a symbolic expression in Sage, its gradient and Hessian can be computed automatically. This allows particular numerical optimization algorithms to be used. They each begin with an initial guess and use an iterative loop to hop to more and more optimal values, terminating once the norm of the gradient passes below a given threshold. Notable algorithms are:

- Conjugate Gradient - essentially a line search for a minima along the gradient direction for each iteration point.
- Newton's method - hopping to the minima of the 2nd order Taylor polynomial for the function at each iteration point (which requires the Hessian to be positive-definite).

If the function is positive-definite at the initial guess and all subsequent points, then Newton's method provides quadratic convergence (referring to the bound on the error).

However, because the energy functional is not globally convex, a bit more care must be taken in computing an approximation of the minimizer.

The tenable solution turned out to be an adaptive combination of Newton's method (preferred) and Conjugate Gradient (good fallback) and Gradient Descent (least preferred). Each iteration was accompanied by a line search of a particular granularity to ensure that the approximation never got worse. An overview of the algorithm is as follows.

- Inputs
  - The functional $\mathcal{L}$, the gradient $\nabla\mathcal{L}$, and Hessian $\nabla^2\mathcal{L}$ (each as a function of $(a_1, \ldots, a_N)$),
  - An initial guess for the optimum (a value in the set of parameters $(a_1, \ldots, a_N)$),
  - An upper bound of tolerance for the gradient, determining one stopping condition.
- Instructions
  1. Set the "current approximation" c to the initial guess.
  2. Compute the gradient $g := \nabla\mathcal{L}(c)$.
  3. If the norm of the gradient is below the tolerance, terminate the algorithm, returning the current approximation.
  4. Compute the Hessian $H := \nabla^2\mathcal{L}(c)$.
  5. Compute the minimizer of the 2nd order Taylor polynomial at the current approximation. This involves solving the linear system $H \cdot v = -g$ for v. The vector v gives the displacement from the current approximation to the optimum of the 2nd order Taylor polynomial.
  6. If $v \cdot H \cdot v > \epsilon$ (for a small $\epsilon > 0$ which depends on the precision of the machine's floating point representation), then the Hessian is positive-definite along v. Newton's Method will be used to proceed. Go to step 9.
  7. If $g \cdot H \cdot g > \epsilon$, then the Hessian is positive-definite along g. Let $v := -g$. The Conjugate Gradient method will be used to proceed. Go to step 9.

8. Otherwise let $v := -\frac{g}{|g|}s$, where $s > 0$ is an optionally-specified Gradient Descent maximum step size. The Gradient Descent method will be used to proceed.

9. Perform a line search for the first element y_i of the sequence

$$\left(c + tv \mid t \in \left\{ \frac{0}{j}, \ldots, \frac{j}{j} \right\} \right)$$

such that $\mathcal{L}(y_i) < \mathcal{L}(y_{i+1})$ or that $i = j$ (the last element). Here, $j \in \mathbb{N}$ is the granularity of the line search. This is to ensure that the algorithm only betters the approximation (though this claim depends on the granularity, which depends on the functional $\mathcal{L}$). Update the current approximation; $c := y_i$. Go to step 2 to continue iterating.

While not mentioned explicitly, to avoid clutter in the description of the algorithm, there is a running counter of the number of iterations. Once this counter exceeds a certain optionally-specified value, the algorithm terminates with failure, to avoid entering certain pathological cases where the algorithm loops indefinitely.

10.3.4 Optimization Using Gradually Increasing Dimension and Gravity

The piecewise linear family of admissible curves has a convenient nested structure with respect to dimension; any given piecewise linear path can produce a piecewise linear path of double the dimension by bisecting each piece, while still representing the same path. This bisection technique was used to start at a low dimensional approximation, where an initial guess for numerical approximation is easy, and gradually crank up the dimension, refining the approximation with each step. The mixed-technique algorithm described above turned out to be necessary to handle this procedure, as in certain cases, the bisection would take the current approximation out of the region of the functional's positive-definiteness.

A second form of successive approximation is done by starting with a low (or zero) gravitational constant – in a high dimensional approximation – and generating successive approximations of the same dimension by slowly cranking up the gravity. Ideally, each step will remain in the positive-definite region around the optimum, and quadratic convergence can in computing the approximate optimum at each step. One small detail is that the (a, G) pair is essentially a projective invariant of the problem; the whole Lagrangian can be divided through by a, preserving the location of the optima. Let $J := \frac{G}{2a}$ as before. This constant is effectively the gravitational constant; the material constant becomes in this case. The gravitational parameter which was varied under the described procedure is J.

10.4 Visualization of Results

10.4.1 Dimension-Increasing Procedure

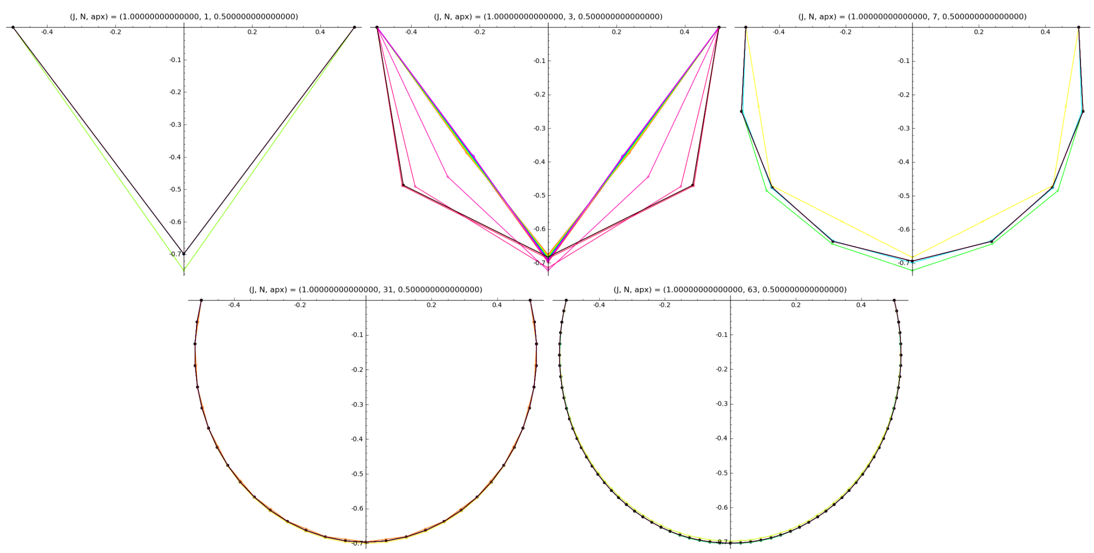


Figure 10.2: Dimension-increasing portion of the procedure. The value of $2J$ is kept fixed at 1, while the dimension of the family of curves varies through the values $N = 1, 3, 7, 15, 31, 63$, from top left to bottom right. All optimizations were well-conditioned and converged quadratically except for the step from $N = 1$ to $N = 3$, where the optimization algorithm had to fall back to the Conjugate Gradient and/or Gradient Descent methods.

10.4.2 Gravity-Increasing Procedure

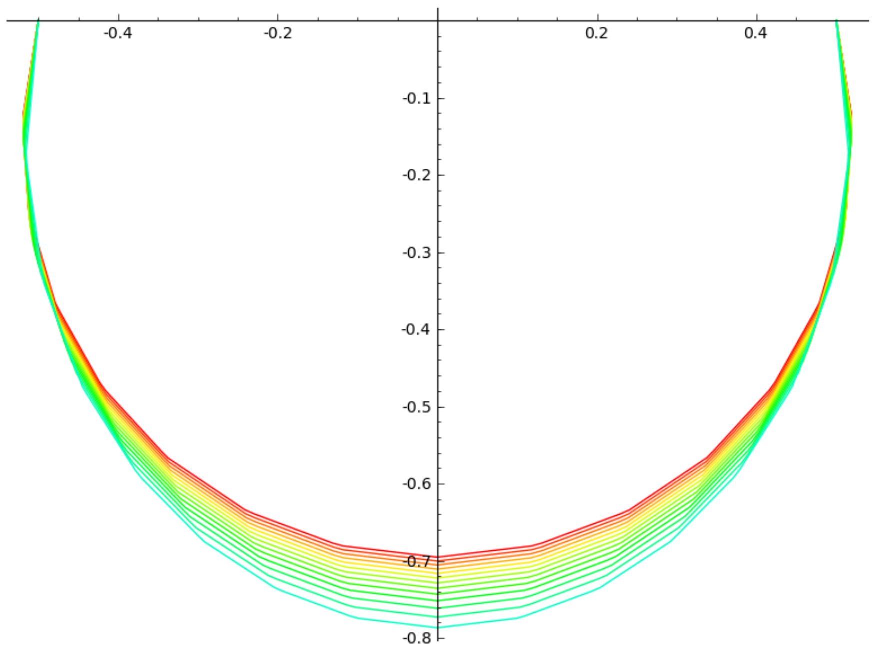


Figure 10.3: Gravity-increasing portion of the procedure. The N parameter here is kept fixed at 15, while the value of $2J$ varies from 1 to 4, in increments of 0.25. This image is this family of approximate solutions in graph coordinates, where the color varies from red ($2J = 1$; low gravity) to blue ($2J = 4$; high gravity).

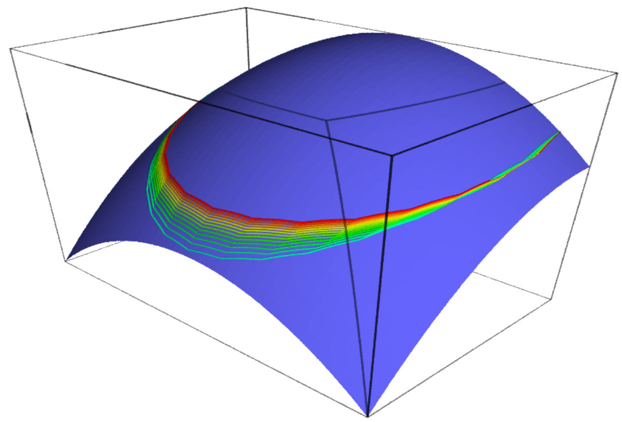


Figure 10.4: This image is the same family of approximate solutions, drawn into the manifold as an embedded graph.

11 Radially Symmetric Dynamic Problem

This section presents a radially symmetric solution of the full, dynamic problem posed originally. Recall that $\dim M = 1$ and $\dim I = 1$; let m be a coordinate on M such that $g_M = dm \otimes dm$, and let t be the standard coordinate on I. By radial symmetry, in graph coordinates, $\phi(m,t) = \rho(t)(\cos m, \sin m)$ for some function $\rho\colon I \to \mathbb{R}$. By calculations made in Section 12.3.7, the Euler-Lagrange [partial differential] equation reduces to the [ordinary differential] equation

$$0 = \left[\rho''(t) + \frac{4\rho(t)\rho'(t)^2}{1+4\rho(t)^2} - 2\rho(t)^{-3}\frac{\rho(t)^4 - J\rho(t)^3 - 1}{1+4\rho(t)^2}\right](\cos m, \sin m).$$

Because $(\cos m, \sin m)$ is a nonvanishing vector field, this implies that

$$0 = \rho''(t) + \frac{4\rho(t)\rho'(t)^2}{1+4\rho(t)^2} - 2\rho(t)^{-3}\frac{\rho(t)^4 - J\rho(t)^3 - 1}{1+4\rho(t)^2}.$$

11.1 Cross Check

If the radius function is constant, this ODE should reduce to the algebraic equation giving the solution to the radially symmetric static problem. Setting $\rho(t) = \rho$

(constant), it follows that $\rho'(t) = 0$ and $\rho''(t) = 0$, and the ODE reduces to

$$0 = -2\rho^{-3}\frac{\rho^4 - J\rho^3 - 1}{1 + 4\rho^2}.$$

Because infinite compression is disallowed, the radius is positive, and therefore this is equivalent to the equation

$$0 = \rho^4 - J\rho^3 - 1,$$

which is indeed the equation derived in the radially symmetric static problem.

11.2 A Conserved Quantity

In the radially symmetric case, the dynamics are given by the solution to an ODE which has a conserved quantity. Define Hamiltonian

$$H := K + W + P.$$

This function is constant along solutions; i.e. $\nabla_{\partial_t} (\overset{*}{\nabla} \phi)^* H = 0$. The proof is a straightforward calculation.

$$\begin{aligned}
&\nabla_{\partial_t} (\overset{*}{\nabla} \phi)^* H \\
&= \nabla_{\partial_t} ((\overset{*}{\nabla} \phi)^* K + (\overset{*}{\nabla} \phi)^* C^* U + (\overset{*}{\nabla} \phi)^* C^* q \ (\overset{*}{\nabla} \phi)^* p_S^* Q) \\
&= \nabla_{\partial_t} \left(\frac{1}{2} |\phi_{,t}|_{g_S}^2 + U \left(|\phi_{,m}|_{g_S}^2 \right) + q \left(|\phi_{,m}|_{g_S}^2 \right) \phi^* Q \right) \\
&= \frac{d}{dt} \left(\frac{1}{2} \left(1 + 4\rho(t)^2 \right) \rho'(t)^2 + \left(\rho(t)^2 - 2 + \rho(t)^{-2} \right) + \rho(t)^{-1} \left(-2J\rho(t)^2 \right) \right) \\
&= \frac{d}{dt} \left(\frac{1}{2} \left(1 + 4\rho(t)^2 \right) \rho'(t)^2 + \rho(t)^2 - 2J\rho(t) - 2 + \rho(t)^{-2} \right) \\
&= \frac{1}{2} 8\rho(t)\rho'(t)\rho'(t)^2 + \frac{1}{2} \left(1 + 4\rho(t)^2 \right) 2\rho'(t)\rho''(t) \\
&\quad + 2\rho(t)\rho'(t) - 2J\rho'(t) - 2\rho(t)^{-3}\rho'(t) \\
&= \left[\left(1 + 4\rho(t)^2 \right) \rho''(t) + 4\rho(t)\rho'(t)^2 + 2\rho(t)^{-3} \left(\rho(t)^4 - J\rho(t)^3 - 1 \right) \right] \rho'(t) \\
&= \left(1 + 4\rho(t)^2 \right) \left[\rho''(t) + \frac{4\rho(t)\rho'(t)^2}{1 + 4\rho(t)^2} + 2\rho(t)^{-3} \frac{\rho(t)^4 - J\rho(t)^3 - 1}{1 + 4\rho(t)^2} \right] \rho'(t) \\
&= 0,
\end{aligned}$$

where the last equality follows because ϕ is assumed to be a solution to the Euler-Lagrange equation, whose reduced form is the vanishing of the second factor.

Thus, with the choice of a fixed energy $H_0 = (\nabla\phi)^* H$, the dynamics reduce to the solution of a first-order ODE. Using part of the the previous computation,

$$\begin{aligned} H_0 &= (\nabla\phi)^* H \\ &= \frac{1}{2}\left(1 + 4\rho(t)^2\right)\rho'(t)^2 + \rho(t)^2 - 2J\rho(t) - 2 + \rho(t)^{-2}. \end{aligned}$$

The phase portrait of this system can be obtained by graphing the level sets of the function

$$G(x, y) := \frac{1}{2}\left(1 + 4x^2\right)y^2 + x^2 - 2Jx - 2 + x^{-2}.$$

The solutions have the form $H_0 = G(\rho(t), \rho'(t))$.

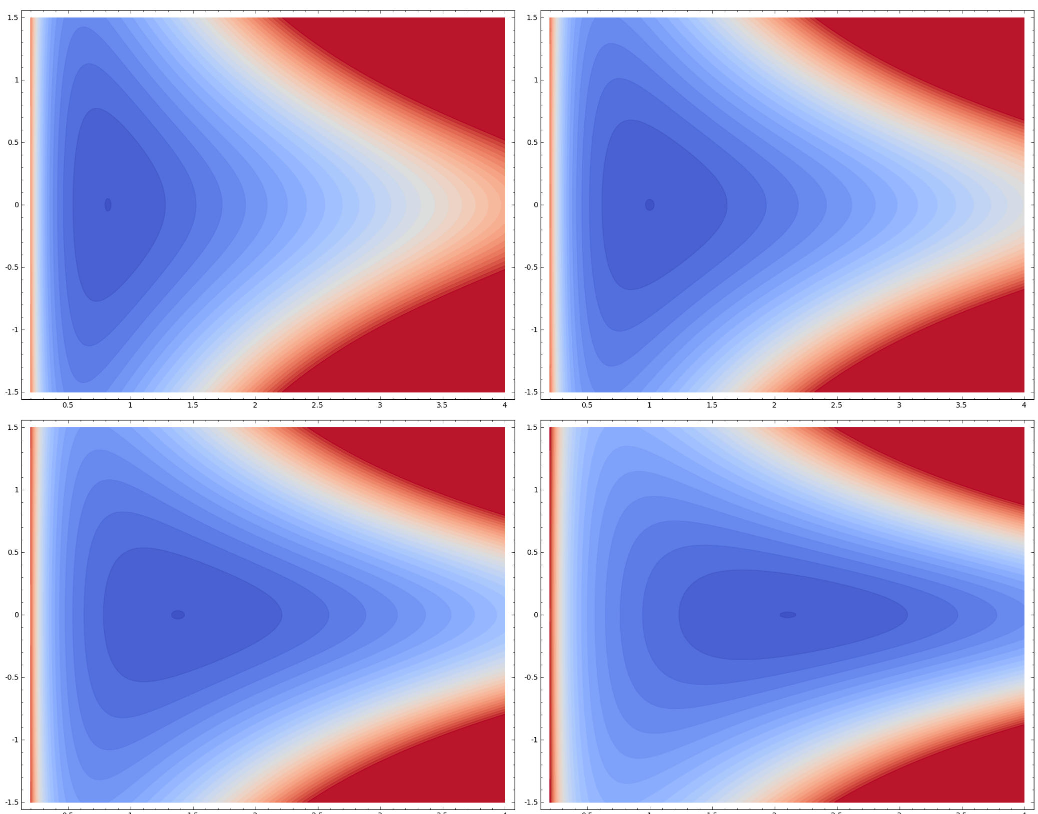


Figure 11.1: Phase portraits of the radially symmetric dynamics. The horizontal axis is ρ, while the vertical axis is ρ'. The level sets of the function are the periodic solutions of the system. Bluer colors indicate lower energy, while redder colors indicate high energy. Top-left: $J = -1$; Top-right: $J = 0$; Bottom-left: $J = 1$, Bottom-right: $J = 2$. It should be noted that the coloring is not coordinated between the diagrams (the same color in different pictures doesn't necessarily represent the same total energy value).

12 Supplemental Results and Calculations

This section contains details of calculations and general results referred to in previous sections. Its placement later in the document is to remove it from the flow of the relevant exposition.

12.1 Radial Vector Field

This section will develop a new result regarding a special vector field which can be defined on any vector bundle. Its discovery was particularly helped along by use of the strongly typed tensor calculus formalism and will be primarily useful in that setting. In particular, because of a pullback property that will be proven in this section, a function of a tensor field which is defined in terms of contractions of said tensor field argument can be defined using the radial vector field as a placeholder for the argument, where the function is evaluated using the pullback by the tensor field. This allows computation of [partial] covariant derivatives to take advantage of the naturality properties regarding the [partial] covariant derivatives of the radial vector field that will be proven in this section. The prime example of such a construction in this exposition is the right Cauchy-Green deformation tensor field (see Section (3.3.3)).

These developments will be done making no type identifications, because part of the result requires a sensitivity in distinguishing types. Some other natural properties of the radial vector field are detailed in [MdL94].

Definition 12.1 (Pullback bundle (category theory formulation))**.** Let N be a manifold, let $\pi_N^E \colon E \to N$ be a [fiber] bundle, and let $\phi \colon M \to N$. Then the pullback [fiber] bundle $\phi^* E$ is defined up to unique isomorphism by the existence of morphisms $\pi_M^{\phi^* E} \colon \phi^* E \to M$ and $\rho_E^{\phi^* E} \colon \phi^* E \to E$ satisfying the following universal property. For any manifold Q and morphisms $b \colon Q \to M$ and $f \colon Q \to E$ where $\phi \circ b = \pi_N^E \circ f$, there exists a unique morphism $\overline{f} \colon Q \to \phi^* E$ such that $\pi_M^{\phi^* E} \circ \overline{f} = b$ and $\rho_E^{\phi^* E} \circ \overline{f} = f$. Call $\overline{f}$ the **lift** of f to $\phi^* E$.

If $\psi \colon L \to M$ and $\phi \colon M \to N$ are manifold morphisms, let

$$\iota_{(\phi \circ \psi)^* E}^{\psi^* \phi^* E} \colon \psi^* \phi^* E \to (\phi \circ \psi)^* E$$

be the canonical isomorphism which facilitates Proposition (19.3).

The standard representation of a pullback bundle $\phi^* E$ is as the embedded submanifold

$$\left\{(e,m) \in E \times M \mid \phi(m) = \pi_N^E(e)\right\} \subset E \times M,$$

in which case the pullback-defining morphisms have the form

$$\pi_M^{\phi^* E} = \mathrm{Pr}_M^{E\times M} |_{\phi^* E},$$
$$\rho_E^{\phi^* E} = \mathrm{Pr}_E^{E\times M} |_{\phi^* E},$$

and the lift $\overline{f} \colon Q \to \phi^* E$ of a morphism $f \colon Q \to M$ has the form

$$\overline{f} = f \times_M \mathrm{Id}_M,$$

and the canonical isomorphism $\iota_{(\phi\circ\psi)^* E}^{\psi^*\phi^* E}$ has the form

$$\iota_{(\phi\circ\psi)^* E}^{\psi^*\phi^* E}((e,m),\ell) = (e,\ell).$$

Definition 12.2 (Type refinement)**.** In the case where $f \in \Gamma_\phi(E)$ (i.e. $f \colon M \to E$ is a section of E along ϕ, meaning that $\pi_N^E \circ f = \phi$), meaning that $Q = M$ and $b = \mathrm{Id}_M$, the lift of f takes the form $\overline{f} \in \Gamma(\phi^* E)$, and $\rho_E^{\phi^* E} \circ \overline{f} = f$, which is a type refinement that enjoys much use in the strongly typed tensor formalism, and in particular, in the following results regarding the canonical radial vector field.

Lemma 12.3 (Existence and uniqueness of vector field with pullback property)**.** *Let $\pi \colon E \to N$ denote a vector bundle. There exists a unique section $r \in \Gamma(\pi^* E)$ such that if $\phi \colon M \to N$ and $\sigma \in \Gamma_\phi(E)$, then*

$$\iota_{(\pi\circ\sigma)^* E}^{\sigma^*\pi^* E} \circ \sigma^* r = \overline{\sigma} \in \Gamma(\phi^* E).$$

Furthermore, representing pullback bundles in the form of embedded submanifolds, $r = \mathrm{Id}_E \times_E \mathrm{Id}_E$.

Proof. For the uniqueness claim, assume that $r_1, r_2 \in \Gamma(\pi^* E)$ both having the pullback property. Note that $\pi \circ \sigma = \phi$. Then

$$\begin{aligned}
0_{\phi^* E} &= \overline{\sigma} - \overline{\sigma} \\
&= \iota^{\sigma^*\pi^*E}_{(\pi\circ\sigma)^*E} \circ \sigma^* r_1 - \iota^{\sigma^*\pi^*E}_{(\pi\circ\sigma)^*E} \circ \sigma^* r_2 && \text{(pullback property)} \\
&= \iota^{\sigma^*\pi^*E}_{(\pi\circ\sigma)^*E} \circ (\sigma^* r_1 - \sigma^* r_2) && \text{(linearity of } \iota^{\sigma^*\pi^*E}_{(\pi\circ\sigma)^*E}\text{)} \\
&= \iota^{\sigma^*\pi^*E}_{(\pi\circ\sigma)^*E} \circ \sigma^* (r_1 - r_2) && \text{(linearity of pullback)} \\
\implies 0_{\sigma^*\pi^*E} &= \left(\iota^{\sigma^*\pi^*E}_{(\pi\circ\sigma)^*E}\right)^{-1} \circ 0_{\phi^* E} && \text{(}\iota^{\sigma^*\pi^*E}_{(\pi\circ\sigma)^*E}\text{ is an isomorphism)} \\
&= \sigma^* (r_1 - r_2) && \text{(from previous equality)} \\
\implies 0_{\pi^* E} &= \rho^{\sigma^*\pi^*E}_{\pi^*E} \circ \sigma^* (r_1 - r_2) && \text{(defining property of pullback)} \\
&= r_1 - r_2 && \text{(from previous equality)} \\
\implies r_1 &= r_2,
\end{aligned}$$

thereby proving uniqueness.

As for the existence claim, it will be shown that if r is defined by $\rho^{\pi^*E}_{E} \circ r = \mathrm{Id}_E$ (and $\pi^{\pi^*E}_{E} \circ r = \mathrm{Id}_E$, but this is implicit in $r \in \Gamma(\pi^* E)$), then r satisfies the claimed pullback property. Let $m \in M$. Then

$$\begin{aligned}
\iota^{\sigma^*\pi^*E}_{(\pi\circ\sigma)^*E} \circ (\sigma^* r)(m) &= \iota^{\sigma^*\pi^*E}_{(\pi\circ\sigma)^*E} ((r \circ \sigma) \times_M \mathrm{Id}_M)(m) \\
&= \iota^{\sigma^*\pi^*E}_{(\pi\circ\sigma)^*E} (r \circ \sigma(m), m) \\
&= \iota^{\sigma^*\pi^*E}_{(\pi\circ\sigma)^*E} ((\mathrm{Id}_E \times \mathrm{Id}_E) \circ \sigma(m), m) \\
&= \iota^{\sigma^*\pi^*E}_{(\pi\circ\sigma)^*E} ((\sigma(m), \sigma(m)), m) \\
&= (\sigma(m), m) \\
&= (\sigma \times_M \mathrm{Id}_M)(m) \\
&= \overline{\sigma}(m),
\end{aligned}$$

and because m was arbitrary, this shows that $\iota^{\sigma^*\pi^*E}_{(\pi\circ\sigma)^*E} \circ \sigma^* r = \overline{\sigma}$, establishing the existence claim. □

Definition 12.4 (Radial vector field)**.** The unique section $r \in \Gamma(\pi^* E)$ in Lemma Lemma 12.3 is called the **radial vector field** on E, or sometimes the **Liouville vector field**.

Lemma 12.5 (Metric-based formulation of radial vector field)**.** *Let $k \in \Gamma(E^* \otimes E^*)$ be a Riemannian metric (positive definite, symmetric, and nondegenerate, with inverse* *metric* $k^{-1} \in \Gamma(E \otimes E)$*) and let* ∇^E *be a linear covariant derivative on* $\pi \colon E \to N$ *that is* k*-compatible, i.e. such that* $\nabla^E k = 0$*. Define quadratic form* $Q \colon E \to \mathbb{R}$ *by* $Q(e) := \frac{1}{2}|e|_k^2 \equiv \frac{1}{2} e \cdot_{E^*} k \cdot_E e$*. Then*

$$Q_{,h} = 0 \quad \textit{and} \quad r = Q_{,v} \cdot_{\pi^* E} \pi^* k^{-1}.$$

Put concisely, the radial vector field can, in a sense, be considered the vertical gradient of a metric-induced quadratic form.

Proof. The vertical derivative $Q_{,v}$ is equivalent to the fiber derivative of Q and can be computed in a straightforward way. Let $(e', e) \in \pi^* E$. In particular, this means that $\pi(e + \epsilon e') = \pi(e)$ for any $\epsilon \in \mathbb{R}$. Then

$$\begin{aligned}
Q_{,v} \cdot_{\pi^* E} (e', e) &= \delta_\epsilon \left(Q(e + \epsilon e') \right) \\
&= \delta_\epsilon \left(\frac{1}{2} (e + \epsilon e') \cdot_{E^*} k \left(\pi(e + \epsilon e') \right) \cdot_E (e + \epsilon e') \right) \\
&= \delta_\epsilon \left(\frac{1}{2} (e + \epsilon e') \cdot_{E^*} k(\pi(e)) \cdot_E (e + \epsilon e') \right) \\
&= \frac{1}{2} e' \cdot_{E^*} k(\pi(e)) \cdot_E e + \frac{1}{2} e \cdot_{E^*} k(\pi(e)) \cdot_E e' \\
&= e \cdot_{E^*} k(\pi(e)) \cdot_E e' \\
&= (e, e) \cdot_{\pi^* E^*} (k \circ \pi(e), e) \cdot_{\pi^* E} (e', e) \\
&= r(e) \cdot_{\pi^* E^*} (\pi^* k)(e) \cdot_{\pi^* E} (e', e) \\
&= (r \cdot_{\pi^* E^*} \pi^* k)(e) \cdot_{\pi^* E} (e', e)
\end{aligned}$$

and because (e', e) was an arbitrary element of $\pi^* E$, this shows that

$$Q_{,v} = r \cdot_{\pi^* E^*} \pi^* k,$$

and by contracting with $\pi^* k^{-1}$,

$$r = Q_{,v} \cdot_{\pi^* E} \pi^* k^{-1},$$

as claimed. □

Lemma 12.6. *Let* k*,* ∇^E*, and* Q *be as in Lemma Lemma 12.5. Then* $Q_{,h} = 0$*.*

Proof. Let $\delta_\epsilon e \in TE$ (i.e. $e\colon \epsilon \to e(\epsilon) \in E$) such that $v \cdot \delta_\epsilon e = 0$, noting that $v \cdot \delta_\epsilon e := \nabla^{(\pi\circ e)^* E}_{\delta_\epsilon} \overline{e}$. Then

$$\begin{aligned}
& Q_{,h} \cdot h \cdot \delta_\epsilon e \\
&= \left(Q_{,h} \cdot h + Q_{,v} \cdot v\right) \cdot \delta_\epsilon e && \text{(since } v \cdot \delta_\epsilon e = 0\text{)} \\
&= \nabla Q \cdot \delta_\epsilon e && \text{(definition of PCDs)} \\
&= \delta_\epsilon \left(Q \circ e\right) && \text{(definition of } \nabla Q\text{)} \\
&= \delta_\epsilon \left(\frac{1}{2} e \cdot_{E^*} \left(k \circ \pi \circ e\right) \cdot_E e\right) && \text{(definition of } Q\text{)} \\
&= \delta_\epsilon \left(\frac{1}{2} \overline{e} \cdot_{(\pi\circ e)^* E^*} \left(\pi \circ e\right)^* k \cdot_{(\pi\circ e)^* E} \overline{e}\right) && \text{(lifting to } (\pi \circ e)^* E\text{)} \\
&= \frac{1}{2} \nabla^{(\pi\circ e)^* E}_{\delta_\epsilon} \overline{e} \cdot_{(\pi\circ e)^* E^*} \left(\pi \circ e\right)^* k \cdot_{(\pi\circ e)^* E} \overline{e} && \text{(product rule)} \\
&\quad + \frac{1}{2} \overline{e} \cdot_{(\pi\circ e)^* E^*} \nabla^{(\pi\circ e)^* (E^* \otimes E^*)}_{\delta_\epsilon} \left(\pi \circ e\right)^* k \cdot_{(\pi\circ e)^* E} \overline{e} \\
&\quad + \frac{1}{2} \overline{e} \cdot_{(\pi\circ e)^* E^*} \left(\pi \circ e\right)^* k \cdot_{(\pi\circ e)^* E} \nabla^{(\pi\circ e)^* E}_{\delta_\epsilon} \overline{e} \\
&= 0 + \frac{1}{2} \overline{e} \cdot_{(\pi\circ e)^* E^*} \nabla^{(\pi\circ e)^* (E^* \otimes E^*)}_{\delta_\epsilon} \left(\pi \circ e\right)^* k \cdot_{(\pi\circ e)^* E} \overline{e} + 0 && \text{(since } \nabla^{(\pi\circ e)^* E}_{\delta_\epsilon} \overline{e} = 0\text{)} \\
&= \frac{1}{2} \overline{e} \cdot_{(\pi\circ e)^* E^*} \left[\left(\pi \circ e\right)^* \nabla^{E^* \otimes E^*} k \cdot \delta_\epsilon \left(\pi \circ e\right)\right] \cdot_{(\pi\circ e)^* E} \overline{e} && \text{(chain rule)} \\
&= \frac{1}{2} \overline{e} \cdot_{(\pi\circ e)^* E^*} \left[\left(\pi \circ e\right)^* 0_{E^* \otimes E^* \otimes T^* N} \cdot \delta_\epsilon \left(\pi \circ e\right)\right] \cdot_{(\pi\circ e)^* E} \overline{e} && \text{(metric compatibility)} \\
&= 0.
\end{aligned}$$

Because $h \cdot \delta_\epsilon e$ can take arbitrary values in $\pi^* TN$, it follows that $Q_{,h} = 0$, as claimed. □

Lemma 12.7. *Let k, ∇^E, and Q be as in Lemma Lemma 12.5. Then $Q_{,vv} = \pi^* k$.*

Proof. The second vertical derivative of Q is analogous to the Hessian along each fiber.

Let $(e', e), (e'', e) \in \pi^* E$. Note that $Q_{,vv} \in \Gamma(\pi^* E^* \otimes \pi^* E^*)$. Then

$$\begin{aligned}
&(e', e) \cdot_{\pi^* E^*} Q_{,vv} \cdot_{\pi^* E} (e'', e) \\
&= \delta_\epsilon \delta_\eta \left(Q \left(e + \epsilon e' + \eta e'' \right) \right) \\
&= \delta_\epsilon \delta_\eta \left(\frac{1}{2} \left(e + \epsilon e' + \eta e'' \right) \cdot_{E^*} k \left(\pi \left(e + \epsilon e' + \eta e'' \right) \right) \cdot_E \left(e + \epsilon e' + \eta e'' \right) \right) \\
&= \delta_\epsilon \delta_\eta \left(\frac{1}{2} \left(e + \epsilon e' + \eta e'' \right) \cdot_{E^*} k \left(\pi (e) \right) \cdot_E \left(e + \epsilon e' + \eta e'' \right) \right) \\
&= \delta_\epsilon \left[\left(e + \epsilon e' \right) \cdot_{E^*} k \left(\pi (e) \right) \cdot_E e'' \right] \\
&= e' \cdot_{E^*} k \left(\pi (e) \right) \cdot_E e'' \\
&= \left(e', e \right) \cdot_{\pi^* E^*} \left(k \circ \pi (e), e \right) \cdot_{\pi^* E} \left(e'', e \right) \\
&= \left(e', e \right) \cdot_{\pi^* E^*} \pi^* k \cdot_{\pi^* E} \left(e'', e \right).
\end{aligned}$$

Because (e', e) and (e'', e) were arbitrary elements of [the same fiber of] $\pi^* E$, it follows that $Q_{,vv} = \pi^* k$, as claimed. □

Lemma 12.8 (Partial covariant derivatives of radial vector field)**.** *Let k and ∇^E be as in Lemma Lemma 12.5. Then $r_{,h} = 0_{\pi^* E \otimes \pi^* T^* N}$ and $r_{,v} = \mathbb{I}_{\pi^* E} \in \Gamma(\pi^* E \otimes \pi^* E^*)$ (the identity tensor field on $\pi^* E$).*

Proof. Let $X \in \pi^* TN$. Using the formula $r = Q_{,v} \cdot \pi^* k^{-1}$ from Lemma Lemma 12.5,

$$\begin{aligned}
r_{,h} \cdot X &= \left(Q_{,v} \cdot \pi^* k^{-1} \right)_{,h} \cdot X \\
&= \left(Q_{,vh} \cdot X \right) \cdot \pi^* k^{-1} + Q_{,v} \cdot \left(\pi^* k^{-1} \right)_{,h} \cdot X && \text{(product rule)} \\
&= \left(X \cdot Q_{,hv} \right) \cdot \pi^* k^{-1} + Q_{,v} \cdot \left(\pi^* k^{-1} \right)_{,h} \cdot X && \text{(since } Q_{,vh} = Q_{,hv}^{(1\,2)}\text{)} \\
&= \left(X \cdot Q_{,hv} \right) \cdot \pi^* k^{-1} + Q_{,v} \cdot 0 \cdot X && \text{(since } \nabla k = 0\text{)} \\
&= \left(X \cdot \left(Q_{,h} \right)_{,v} \right) \cdot \pi^* k^{-1} && \text{(iterated PCDs)} \\
&= \left(X \cdot (0)_{,v} \right) \cdot \pi^* k^{-1} && \text{(by 12.6)} \\
&= 0,
\end{aligned}$$

and because X was an arbitrary element of $\pi^* TN$, it follows that

$$r_{,h} = 0 \in \Gamma(\pi^* E \otimes \pi^* T^* N).$$

For the vertical PCD, Let $Y \in \pi^* E$. Then, using the formula $r = Q_{,v} \cdot \pi^* k^{-1}$ again,

$$\begin{aligned}
r_{,v} \cdot Y &= \left(Q_{,v} \cdot \pi^* k^{-1}\right)_{,v} \cdot Y \\
&= \left(Q_{,vv} \cdot Y\right) \cdot \pi^* k^{-1} + Q_{,v} \cdot \left(\pi^* k^{-1}\right)_{,v} \cdot Y && \text{(product rule)} \\
&= \left(Q_{,vv} \cdot Y\right) \cdot \pi^* k^{-1} + Q_{,v} \cdot 0 \cdot Y && \text{(since } \nabla k = 0\text{)} \\
&= \pi^* k^{-1} \cdot Q_{,vv} \cdot Y && \text{(symmetry of } k^{-1}\text{)} \\
&= \pi^* k^{-1} \cdot \pi^* k \cdot Y && \text{(by 12.7)} \\
&= \mathbb{I}_{\pi^* E} \cdot Y && \text{(definition of metric inverse).}
\end{aligned}$$

Because Y was arbitrary, this implies that $r_{,v} = \mathbb{I}_{\pi^* E}$, as claimed. □

12.2 Additional Lemmas

Lemma 12.9. *Let $\gamma \in C^2([t_0, t_1], S)$ be a regular curve (i.e. $|\gamma'|$ is nonvanishing). Then $\nabla_{\frac{d}{dt}} \gamma' = f\,\gamma'$ for some function $f \in C^0([t_0, t_1], \mathbb{R})$ implies that $f = \frac{d}{dt}(\log |\gamma'|)$.*

Proof. Let $\nabla_{\frac{d}{dt}} \gamma' = f\,\gamma'$ for some function f as described. Then

$$\frac{d}{dt}\left|\gamma'\right| = \frac{d}{dt}\left(\gamma' \cdot h \cdot \gamma'\right)^{1/2} = \frac{1}{2}\frac{2\gamma' \cdot h \cdot \nabla_{\frac{d}{dt}}\gamma'}{\left(\gamma' \cdot h \cdot \gamma'\right)^{1/2}} = \frac{\gamma' \cdot h \cdot (f\,\gamma')}{|\gamma'|} = f\frac{|\gamma'|^2}{|\gamma'|} = f\left|\gamma'\right|,$$

and therefore

$$f = \frac{\frac{d}{dt}\left|\gamma'\right|}{|\gamma'|} = \frac{d}{dt}\left(\log\left|\gamma'\right|\right),$$

as desired. □

Lemma 12.10. *Let $\gamma \in C^2([t_0, t_1], S)$ be a regular curve (i.e. $|\gamma'|$ is nonvanishing). Then $\nabla_{\frac{d}{dt}} \gamma' = f\,\gamma'$ for some function $f \in C^0([t_0, t_1], \mathbb{R})$ if and only if γ is the reparameterization of a geodesic having the same image.*

Proof. Let $s(t) := \int_{t_0}^{t} |\gamma'(\tau)|\, d\tau$ be the arclength function, noting that $s' = |\gamma'|$. Let $\phi = \gamma \circ s^{-1}$. Then

$$\phi' \circ s = \left(\left(\gamma' \circ s^{-1}\right)\left(s^{-1}\right)'\right) \circ s = \frac{\gamma' \circ s^{-1}}{s' \circ s^{-1}} \circ s = \frac{\gamma'}{s'},$$

and

$$\begin{aligned}
\nabla_{\frac{d}{dt}}\gamma' - \frac{s''}{s'}\gamma' &= \nabla_{\frac{d}{dt}}(\phi\circ s)' - \frac{s''}{s'}\gamma' \\
&= \left(\nabla_{\frac{d}{ds}}\phi'\circ s\right)\left(s'\right)^2 + \left(\phi'\circ s\right)s'' - \frac{s''}{s'}\gamma' \\
&= \left(\nabla_{\frac{d}{ds}}\phi'\circ s\right)\left(s'\right)^2 + \left(\frac{\gamma'\circ s^{-1}}{s'\circ s^{-1}}\circ s\right)s'' - \frac{s''}{s'}\gamma' \\
&= \left(\nabla_{\frac{d}{ds}}\phi'\circ s\right)\left(s'\right)^2 + \frac{s''}{s'}\gamma' - \frac{s''}{s'}\gamma' \\
&= \left(\nabla_{\frac{d}{ds}}\phi'\circ s\right)\left(s'\right)^2.
\end{aligned}$$

For the " $\implies$ " direction, assume that $\nabla_{\frac{d}{dt}}\gamma' = f\,\gamma'$ as described. Then by Lemma Lemma 12.9, $f = \frac{d}{dt}(\log|\gamma'|)$, and therefore $f = s''/s'$. By the calculations above,

$$\begin{aligned}
0 &= \nabla_{\frac{d}{dt}}\gamma' - f\,\gamma' \\
&= \left(\nabla_{\frac{d}{ds}}\phi'\circ s\right)\left(s'\right)^2 && \text{(by above calculations)} \\
\implies 0 &= \nabla_{\frac{d}{ds}}\phi'\circ s && (s' \text{ is nonvanishing}) \\
\implies 0 &= \nabla_{\frac{d}{ds}}\phi' && \text{(i.e. } \phi \text{ is a geodesic)},
\end{aligned}$$

showing that γ is the reparameterization of a geodesic with the same image, as desired.

For the "$\impliedby$" direction, assume that γ is the reparameterization of a geodesic ϕ having the same image. It follows, because γ is regular, that $\gamma = \phi\circ s$. Then

$$\begin{aligned}
0 &= \nabla_{\frac{d}{ds}}\phi' && (\phi \text{ is a geodesic}) \\
\implies 0 &= \left(\nabla_{\frac{d}{ds}}\phi'\circ s\right)\left(s'\right)^2 \\
&= \nabla_{\frac{d}{dt}}\gamma' - \frac{s''}{s'}\gamma',
\end{aligned}$$

so with $f := s''/s'$, it follows that $\nabla_{\frac{d}{dt}}\gamma' = f\,\gamma'$, as desired. □

12.3 Calculations

Summarizing the notations and conventions from Section Section 5, the relevant spaces, bundles, bundle projection maps, and canonical structures are

$$(M, g_M) \qquad \text{(material manifold)},$$

$$(S, g_S) \qquad \text{(spatial manifold)},$$

$$N := M \times I \qquad \text{(time-parameterized material manifold)},$$

$$E := TS \otimes T^*N \qquad \text{(time-parameterized local deformation)},$$

$$\pi\colon E \to S \times N \qquad \text{(bundle projection for } E\text{)},$$

$$p_S := \Pr_S^{S\times N} \circ \pi\colon E \to S \qquad \text{(spatial component of bundle proj.)},$$

$$p_N := \Pr_N^{S\times N} \circ \pi\colon E \to N \qquad \text{(material/temporal component of bundle proj.)},$$

$$r \in \Gamma(\pi^* E) \qquad \text{(canonical radial vector field)},$$

$$g_S \boxtimes (\partial_m \otimes \partial_m) \in \Gamma(E^* \otimes E^*), \qquad \text{(material summand of metric induced on } E\text{)},$$

$$g_S \boxtimes (\partial_t \otimes \partial_t) \in \Gamma(E^* \otimes E^*), \qquad \text{(temporal summand of metric induced on } E\text{)}.$$

The embedding map and related deformation tensors are

$$\phi\colon N \to S \qquad \text{(time-parameterized material embedding)},$$

$$\overline{\phi} := \phi \times_{\mathrm{Id}_N} \mathrm{Id}_N\colon N \to S \times N \qquad \text{(type refinement of } \phi\text{)},$$

$$\nabla\, \phi \in \Gamma_{\overline{\phi}}(E) \qquad \text{("primitive" deformation gradient)},$$

$$\overline{\nabla\, \phi} \in \Gamma\left(\overline{\phi}^* E\right) \qquad \text{(deformation gradient)},$$

$$B := r \cdot_{\pi^* E^*} \pi^* (g_S \boxtimes (\partial_t \otimes \partial_t)) \cdot_{\pi^* E} r \qquad \text{(temporal quadratic form on } \pi^* E\text{)},$$

$$C := r \cdot_{\pi^* E^*} \pi^* (g_S \boxtimes (\partial_m \otimes \partial_m)) \cdot_{\pi^* E} r \qquad \text{(material quadratic form on } \pi^* E\text{)},$$

$$\frac{1}{2} B \circ \nabla\, \phi \equiv (\nabla\, \phi)^* \left(\frac{1}{2} B\right) \qquad \text{(local kinetic energy of } \phi\text{)},$$

$$C \circ \nabla\, \phi \equiv (\nabla\, \phi)^* C \qquad \text{(local squared stretch of } \phi\text{)}.$$

Finally, the energy functions and other helper functions are

$$
\begin{aligned}
Q\colon S &\to \mathbb{R} && \text{(potential energy density in spatial manifold),} \\
q\colon \mathbb{R} &\to \mathbb{R} && \text{(energy-density-to-energy conversion helper)} \\
x &\mapsto x^{-1/2} \\
U\colon \mathbb{R}^{+} &\to \mathbb{R} && \text{(hyperelastic material stored energy function).}
\end{aligned}
$$

Now the kinetic energy, stored energy, and potential energy terms can be defined, so that the Lagrangian can be constructed. The kinetic energy is $K := \frac{1}{2}B\colon E \to \mathbb{R}$ (the density of the string is considered to be unit). The stored energy term is $W := U\left(|\phi_{,m}|^2\right)$, which can be constructed as $W := U \circ C\colon E \to \mathbb{R}$, and written in the pullback formalism as $W := C^*U$. The potential energy term is induced by the potential energy density function by the string embedding: $P\left(|\phi_{,m}|\right) := |\phi_{,m}|^{-1} Q \circ \phi$, which can be written without the parameter as $P := C^* q\, p_S^* Q\colon E \to \mathbb{R}$.

The Lagrangian is the difference of the kinetic energy and all potential energy terms.

$$
L := K - (W + P)\,.
$$

12.3.1 Euler-Lagrange Equation

The general form of the Euler-Lagrange equation for $L\colon E \to \mathbb{R}$ and solution map $\phi\colon N \to S$ is defined in terms of partial covariant derivatives which require a few more structures to be defined. Let $h := \overline{\nabla \pi} \in \Gamma\left(\pi^* T\left(S \times N\right) \otimes T^*E\right)$, and let $v \in \Gamma\left(\pi^* E \otimes T^*E\right)$ be defined by the linear covariant derivative ∇^E tensorially induced by the Levi-Civita linear covariant derivatives ∇^{TS} and ∇^{TN};

$$
v \cdot \delta_\epsilon e := \nabla_{\delta_\epsilon}^{(\pi\circ e)^* E}\overline{e},
$$

where $e\colon I \to E$ is any curve representing an arbitrary tangent vector $\delta_\epsilon e \in TE$. The definition of v can be summarized by saying that it replaces an ordinary derivative with a covariant one. Finally, with

$$
\sigma := \overline{\nabla\, p_S} \in \Gamma\left(p_S^* TS \otimes T^*E\right)
$$

and $\mu \in \Gamma\left(p_N^* TN \otimes T^*E\right)$, noting that $\sigma \oplus \mu = h$, it follows that $\sigma \oplus \mu \oplus v \in \Gamma\left(\pi^*\left(TS \oplus TN \oplus E\right) \otimes T^*E\right)$ is a vector bundle isomorphism, and allows the necessary partial covariant derivatives to be defined. With $\nabla L \in \Gamma\left(T^{*}E\right)$ denoting the differential of the function L on E, $L_{,\sigma} \in \Gamma\left(p_{S}^{*}T^{*}S\right)$, $L_{,\mu} \in \Gamma\left(p_{N}^{*}T^{*}N\right)$, and $L_{,v} \in \Gamma\left(\pi^{*}E^{*}\right)$ are defined via the equation

$$\nabla L = L_{,\sigma} \cdot_{p_{S}^{*}TS} \sigma + L_{,\mu} \cdot_{p_{N}^{*}TN} \mu + L_{,v} \cdot_{\pi^{*}E} v.$$

Finally, the Euler-Lagrange equation can be written.

$$0 = \left(\not\nabla\, \phi\right)^{*} L_{,\sigma} - \operatorname{Div}_{N}\left(\left(\not\nabla\, \phi\right)^{*} L_{,v}\right) \text{ on } N,$$

12.3.2 Calculating the Particular Euler-Lagrange Equation

Dense calculations follow. Let $X \in TE$. A fact that won't be proven here is that $r_{,h} = 0$ (so $r_{,\sigma} = 0 \in \Gamma\left(\pi^{*}E \otimes p_{S}^{*}T^{*}S\right)$) and $r_{,v} = \mathbb{I}_{\pi^{*}E} \in \Gamma\left(\pi^{*}E \otimes \pi^{*}E^{*}\right)$.

$$\begin{aligned}
& B_{,\sigma} \cdot_{p_{S}^{*}TS} \sigma \cdot_{TE} X \\
={}& \left(r \cdot_{\pi^{*}E^{*}} \pi^{*}\left(g_{S} \boxtimes \left(\partial_{t} \otimes \partial_{t}\right)\right) \cdot_{\pi^{*}E} r\right)_{,\sigma} \cdot_{p_{S}^{*}TS} \sigma \cdot_{TE} X \\
={}& \left[r_{,\sigma} \cdot_{p_{S}^{*}TS} \sigma \cdot_{TE} X\right] \cdot_{\pi^{*}E^{*}} \pi^{*}\left(g_{S} \boxtimes \left(\partial_{t} \otimes \partial_{t}\right)\right) \cdot_{\pi^{*}E} r \\
& + r \cdot_{\pi^{*}E^{*}} \left[\left(\pi^{*}\left(g_{S} \boxtimes \left(\partial_{t} \otimes \partial_{t}\right)\right)\right)_{,\sigma} \cdot_{p_{S}^{*}TS} \sigma \cdot_{TE} X\right] \cdot_{\pi^{*}E} r \\
& + r \cdot_{\pi^{*}E^{*}} \pi^{*}\left(g_{S} \boxtimes \left(\partial_{t} \otimes \partial_{t}\right)\right) \cdot_{\pi^{*}E} \left[r_{,\sigma} \cdot_{p_{S}^{*}TS} \sigma \cdot_{TE} X\right] \\
={}& \left[0 \cdot_{p_{S}^{*}TS} \sigma \cdot_{TE} X\right] \cdot_{\pi^{*}E^{*}} \pi^{*}\left(g_{S} \boxtimes \left(\partial_{t} \otimes \partial_{t}\right)\right) \cdot_{\pi^{*}E} r && \left(r_{,\sigma} = 0\right) \\
& + r \cdot_{\pi^{*}E^{*}} \left[\pi^{*}\nabla\left(g_{S} \boxtimes \left(\partial_{t} \otimes \partial_{t}\right)\right) \cdot_{\pi^{*}T(S \times N)} \pi_{,\sigma} \cdot_{p_{S}^{*}TS} \sigma \cdot_{TE} X\right] \cdot_{\pi^{*}E} r \\
& + r \cdot_{\pi^{*}E^{*}} \pi^{*}\left(g_{S} \boxtimes \left(\partial_{t} \otimes \partial_{t}\right)\right) \cdot_{\pi^{*}E} \left[0 \cdot_{p_{S}^{*}TS} \sigma \cdot_{TE} X\right] \\
={}& r \cdot_{\pi^{*}E^{*}} \left[\pi^{*}0 \cdot \pi_{,\sigma} \cdot_{p_{S}^{*}TS} \sigma \cdot_{TE} X\right] \cdot_{\pi^{*}E} r \\
={}& 0.
\end{aligned}$$

The second to last equality follows because the fields g_{S} and $\partial_{t} \otimes \partial_{t}$ are parallel, and therefore $\nabla\left(g_{S} \boxtimes \left(\partial_{t} \otimes \partial_{t}\right)\right) = 0$. Because $\sigma \cdot_{TE} X$ can take arbitrary values in $p_{S}^{*}TS$, it follows that

$$\begin{aligned}
& B_{,\sigma} = 0, \\
& C_{,\sigma} = 0 && \text{(by lexically analogous calculation).}
\end{aligned}$$

For the vertical derivative,

$$\begin{aligned}
&B_{,v} \cdot_{\pi^* E} v \cdot_{TE} X \\
&= \left[r_{,v} \cdot_{\pi^* E} v \cdot_{TE} X\right] \cdot_{\pi^* E^*} \pi^* \left(g_S \boxtimes \left(\partial_t \otimes \partial_t\right)\right) \cdot_{\pi^* E} r \\
&\quad + r \cdot_{\pi^* E^*} \left[\left(\pi^* \left(g_S \boxtimes \left(\partial_t \otimes \partial_t\right)\right)\right)_{,v} \cdot_{\pi^* E} v \cdot_{TE} X\right] \cdot_{\pi^* E} r \\
&\quad + r \cdot_{\pi^* E^*} \pi^* \left(g_S \boxtimes \left(\partial_t \otimes \partial_t\right)\right) \cdot_{\pi^* E} \left[r_{,v} \cdot_{\pi^* E} v \cdot_{TE} X\right] \\
&= \left[\mathbb{I}_{\pi^* E} \cdot_{\pi^* E} v \cdot_{TE} X\right] \cdot_{\pi^* E^*} \pi^* \left(g_S \boxtimes \left(\partial_t \otimes \partial_t\right)\right) \cdot_{\pi^* E} r \\
&\quad + r \cdot_{\pi^* E^*} \left[\pi^* \nabla \left(g_S \boxtimes \left(\partial_t \otimes \partial_t\right)\right) \cdot_{\pi^* T(S \times N)} \pi_{,v} \cdot_{\pi^* E} v \cdot_{TE} X\right] \cdot_{\pi^* E} r \\
&\quad + r \cdot_{\pi^* E^*} \pi^* \left(g_S \boxtimes \left(\partial_t \otimes \partial_t\right)\right) \cdot_{\pi^* E} \left[\mathbb{I}_{\pi^* E} \cdot_{\pi^* E} v \cdot_{TE} X\right] \\
&= r \cdot_{\pi^* E^*} \pi^* \left(g_S \boxtimes \left(\partial_t \otimes \partial_t\right)\right) \cdot_{\pi^* E} \left[\mathbb{I}_{\pi^* E} \cdot_{\pi^* E} v \cdot_{TE} X\right] \\
&\quad + r \cdot_{\pi^* E^*} \left[\pi^* 0 \cdot_{\pi^* T(S \times N)} \pi_{,v} \cdot_{\pi^* E} v \cdot_{TE} X\right] \cdot_{\pi^* E} r \\
&\quad + r \cdot_{\pi^* E^*} \pi^* \left(g_S \boxtimes \left(\partial_t \otimes \partial_t\right)\right) \cdot_{\pi^* E} \left[\mathbb{I}_{\pi^* E} \cdot_{\pi^* E} v \cdot_{TE} X\right] \\
&= 2r \cdot_{\pi^* E^*} \pi^* \left(g_S \boxtimes \left(\partial_t \otimes \partial_t\right)\right) \cdot_{\pi^* E} v \cdot_{TE} X.
\end{aligned}$$

Because $v \cdot_{TE} X$ can take arbitrary values in $\pi^* E$, it follows that

$$\begin{aligned}
B_{,v} &= 2r \cdot_{\pi^* E^*} \pi^* \left(g_S \boxtimes \left(\partial_t \otimes \partial_t\right)\right), \\
C_{,v} &= 2r \cdot_{\pi^* E^*} \pi^* \left(g_S \boxtimes \left(\partial_m \otimes \partial_m\right)\right) \qquad \text{(by lexically analogous calculation).}
\end{aligned}$$

It will be necessary to calculate $\left(p_S\right)_{,\sigma}$ and $\left(p_S\right)_{,v}$. Let $Y \in TE$ such that $\mu \cdot Y = 0$ and $v \cdot Y = 0$. Then

$$\begin{aligned}
\left(p_S\right)_{,\sigma} \cdot \sigma \cdot Y &= \left(\left(p_S\right)_{,\sigma} \cdot \sigma + \left(p_S\right)_{,\mu} \cdot \mu + \left(p_S\right)_{,v} \cdot v\right) \cdot Y \\
&= \overline{\nabla p_S} \cdot Y \\
&= \sigma \cdot Y \\
&= \mathbb{I}_{p_S^* TS} \cdot \sigma \cdot Y,
\end{aligned}$$

and because $\sigma \cdot Y$ can take arbitrary values in $p_S^* TS$, it follows that

$$\left(p_S\right)_{,\sigma} = \mathbb{I}_{p_S^* TS}.$$

Similarly, let $\delta_\epsilon \left(e + \epsilon e'\right) \in TE$, meaning that $\delta_\epsilon \left(e + \epsilon e'\right)$ lies within the kernels of σ and μ, i.e. $\sigma \cdot \delta_\epsilon (e + \epsilon e') = 0$ and $\mu \cdot \delta_\epsilon (e + \epsilon e') = 0$. Then

$$
\begin{aligned}
& (p_S)_{,v} \cdot v \cdot \delta_\epsilon \left(e + \epsilon e'\right) \\
= & \left((p_S)_{,\sigma} \cdot \sigma + (p_S)_{,\mu} \cdot \mu + (p_S)_{,v} \cdot v\right) \cdot \delta_\epsilon \left(e + \epsilon e'\right) \\
= & \overline{\nabla p_S} \cdot \delta_\epsilon \left(e + \epsilon e'\right) \\
= & \overline{\delta_\epsilon \left(p_S \left(e + \epsilon e'\right)\right)} \\
= & \overline{\delta_\epsilon \left(\Pr_S^{S \times N} \circ \pi \left(e + \epsilon e'\right)\right)} \\
= & \overline{\delta_\epsilon \left(\Pr_S^{S \times N} \circ \pi \left(e\right)\right)} && (e + \epsilon e' \text{lies in a single fiber}) \\
= & \, 0,
\end{aligned}
$$

and because $v \cdot \delta_\epsilon (e + \epsilon e')$ can take arbitrary values in $\pi^* E$, it follows that

$$(p_S)_{,v} = 0.$$

Now to calculate $L_{,\sigma}$. Note that $q'(x) = -\frac{1}{2} x^{-3/2} = -\frac{1}{2} q(x)^3$, so $q' = -\frac{1}{2} q^3$. Then

$$
\begin{aligned}
K_{,\sigma} &= \left(\frac{1}{2} B\right)_{,\sigma} = 0, \\
W_{,\sigma} &= (C^* U)_{,\sigma} = C^* U' \, C_{,\sigma} = C^* U' \, 0 = 0,
\end{aligned}
$$

$$
\begin{aligned}
P_{,\sigma} \cdot \sigma \cdot X &= (C^* q \, p_S^* Q)_{,\sigma} \cdot \sigma \cdot X \\
&= p_S^* Q \, (C^* q)_{,\sigma} \cdot \sigma \cdot X + C^* q \, (p_S^* Q)_{,\sigma} \cdot \sigma \cdot X \\
&= p_S^* Q \, C^* q' \, C_{,\sigma} \cdot \sigma \cdot X + C^* q \, p_S^* \nabla Q \cdot (p_S)_{,\sigma} \cdot \sigma \cdot X \\
&= p_S^* Q \, C^* q' \, 0 \cdot \sigma \cdot X + C^* q \, p_S^* \nabla Q \cdot \mathbb{I}_{p_S^* TS} \cdot \sigma \cdot X \\
&= C^* q \, p_S^* \nabla Q \cdot \sigma \cdot X \\
\implies P_{,\sigma} &= C^* q \, p_S^* \nabla Q,
\end{aligned}
$$

and therefore

$$L_{,\sigma} = K_{,\sigma} - W_{,\sigma} - P_{,\sigma} = 0 - 0 - C^* q \, p_S^* \nabla Q = -C^* q \, p_S^* \nabla Q.$$

Now to calculate $L_{,v}$;

$$
\begin{aligned}
K_{,v} &= \left(\frac{1}{2}B\right)_{,v} \\
&= \frac{1}{2}\,2r \cdot_{\pi^* E^*} \pi^* \left(g_S \otimes (\partial_t \otimes \partial_t)\right) \\
&= r \cdot_{\pi^* E^*} \pi^* \left(g_S \otimes (\partial_t \otimes \partial_t)\right), \\
W_{,v} &= \left(C^* U\right)_{,v} \\
&= C^* U' \, C_{,v} \\
&= 2C^* U' \, r \cdot_{\pi^* E^*} \pi^* \left(g_S \boxtimes (\partial_m \otimes \partial_m)\right), \\
P_{,v} &= \left(C^* q \, p_S^* Q\right)_{,v} \\
&= p_S^* Q \, \left(C^* q\right)_{,v} + C^* q \, \left(p_S^* Q\right)_{,v} \\
&= p_S^* Q \, C^* q' \, C_{,v} + C^* q \, p_S^* \nabla Q \cdot \left(p_S\right)_{,v} \\
&= p_S^* Q \, C^* \left(-\frac{1}{2} q^3\right) \, 2r \cdot_{\pi^* E^*} \pi^* \left(g_S \boxtimes (\partial_m \otimes \partial_m)\right) + C^* q \, p_S^* \nabla Q \cdot 0 \\
&= -p_S^* Q \, C^* q^3 \, r \cdot_{\pi^* E^*} \pi^* \left(g_S \boxtimes (\partial_m \otimes \partial_m)\right),
\end{aligned}
$$

and therefore

$$
\begin{aligned}
L_{,v} &= K_{,v} - W_{,v} - P_{,v} \\
&= r \cdot_{\pi^* E^*} \pi^* \left(g_S \boxtimes (\partial_t \otimes \partial_t)\right) - 2C^* U' \, r \cdot_{\pi^* E^*} \pi^* \left(g_S \boxtimes (\partial_m \otimes \partial_m)\right) \\
&\quad - \left(-p_S^* Q \, C^* q^3 \, r \cdot_{\pi^* E^*} \pi^* \left(g_S \boxtimes (\partial_m \otimes \partial_m)\right)\right) \\
&= r \cdot_{\pi^* E^*} \pi^* \left(g_S \boxtimes (\partial_t \otimes \partial_t)\right) + \left(p_S^* Q \, C^* q^3 - 2C^* U'\right) \, r \cdot_{\pi^* E^*} \pi^* \left(g_S \boxtimes (\partial_m \otimes \partial_m)\right).
\end{aligned}
$$

Now to calculate the pullbacks $(\boldsymbol{\nabla}\,\phi)^* \, L_{,\sigma}$ and $(\boldsymbol{\nabla}\,\phi)^* \, L_{,v}$ (these play the role, in the pullback formalism, of the compositions of $\boldsymbol{\nabla}\ \phi$ into $L_{,\sigma}$ and $L_{,v}$). Here, the natural pullback property of the radial vector field will be used; $(\boldsymbol{\nabla}\,\phi)^* \, r = \overline{\boldsymbol{\nabla}\,\phi}$. The relevant quantities are

$$
\begin{aligned}
C \circ \nabla\!\!\!\!\nabla\, \phi &= |\phi_{,m}|^2_{g_S}\,, \\
p_S \circ \nabla\!\!\!\!\nabla\, \phi &= \phi, \\
\pi \circ \nabla\!\!\!\!\nabla\, \phi &= \overline{\phi}, \\
(\nabla\!\!\!\!\nabla\, \phi)^* \, p_S^* Q &= (p_S \circ \nabla\!\!\!\!\nabla\, \phi)^* \, Q = \phi^* Q, \\
(\nabla\!\!\!\!\nabla\, \phi)^* \, C^* q^3 &= q^3 \circ C \circ \nabla\!\!\!\!\nabla\, \phi = q^3 \left(|\phi_{,m}|^2 \right) = |\phi_{,m}|^{-3}\,, \\
(\nabla\!\!\!\!\nabla\, \phi)^* \, C^* U' &= U' \circ C \circ \nabla\!\!\!\!\nabla\, \phi = U' \left(|\phi_{,m}|^2 \right), \\
(\nabla\!\!\!\!\nabla\, \phi)^* \, L_{,\sigma} &= (\nabla\!\!\!\!\nabla\, \phi)^* \left(-C^* q \, p_S^* \nabla Q \right) \\
&= -\, (\nabla\!\!\!\!\nabla\, \phi)^* \, C^* q \; (\nabla\!\!\!\!\nabla\, \phi)^* \, p_S^* \nabla Q \\
&= -q \circ C \circ \nabla\!\!\!\!\nabla\, \phi \; (p_S \circ \nabla\!\!\!\!\nabla\, \phi)^* \, \nabla Q \\
&= -q \left(|\phi_{,m}|^2 \right) \; \phi^* \nabla Q \\
&= -\, |\phi_{,m}|^{-1} \; \phi^* \nabla Q,
\end{aligned}
$$

$$
\begin{aligned}
& (\nabla\!\!\!\!\nabla\, \phi)^* \, L_{,v} \\
= \; & (\nabla\!\!\!\!\nabla\, \phi)^* \left(r \cdot_{\pi^* E^*} \pi^* \left(g_S \boxtimes (\partial_t \otimes \partial_t) \right) \right) \\
& + (\nabla\!\!\!\!\nabla\, \phi)^* \left(p_S^* Q \, C^* q^3 - 2 C^* U' \right) \, r \cdot_{\pi^* E^*} \pi^* \left(g_S \boxtimes (\partial_m \otimes \partial_m) \right) \\
= \; & (\nabla\!\!\!\!\nabla\, \phi)^* \, r \cdot_{(\nabla\!\!\!\!\nabla\, \phi)^* \pi^* E^*} (\nabla\!\!\!\!\nabla\, \phi)^* \, \pi^* \left(g_S \boxtimes (\partial_t \otimes \partial_t) \right) \\
& + (\nabla\!\!\!\!\nabla\, \phi)^* \left(p_S^* Q \, C^* q^3 - 2 C^* U' \right) \, (\nabla\!\!\!\!\nabla\, \phi)^* \, r \cdot_{(\nabla\!\!\!\!\nabla\, \phi)^* \pi^* E^*} (\nabla\!\!\!\!\nabla\, \phi)^* \, \pi^* \left(g_S \boxtimes (\partial_m \otimes \partial_m) \right) \\
= \; & \overline{\nabla\!\!\!\!\nabla\, \phi} \cdot_{\overline{\phi}^* E^*} (\pi \circ \nabla\!\!\!\!\nabla\, \phi)^* \left(g_S \boxtimes (\partial_t \otimes \partial_t) \right) \\
& + \left(\phi^* Q \; |\phi_{,m}|^{-3} - 2 U' \left(|\phi_{,m}|^2 \right) \right) \, \overline{\nabla\!\!\!\!\nabla\, \phi} \cdot_{\overline{\phi}^* E^*} (\pi \circ \nabla\!\!\!\!\nabla\, \phi)^* \left(g_S \boxtimes (\partial_m \otimes \partial_m) \right) \\
= \; & \overline{\nabla\!\!\!\!\nabla\, \phi} \cdot_{\overline{\phi}^* E^*} \overline{\phi}^* \left(g_S \boxtimes (\partial_t \otimes \partial_t) \right) \\
& + \left(\phi^* Q \; |\phi_{,m}|^{-3} - 2 U' \left(|\phi_{,m}|^2 \right) \right) \, \overline{\nabla\!\!\!\!\nabla\, \phi} \cdot_{\overline{\phi}^* E^*} \overline{\phi}^* \left(g_S \boxtimes (\partial_m \otimes \partial_m) \right) \\
= \; & \left(\overline{\nabla\!\!\!\!\nabla\, \phi} \cdot_{TN} \partial_t \right) \cdot_{\phi^* T^* S} \phi^* g_S \otimes \partial_t \\
& + \left(\phi^* Q \; |\phi_{,m}|^{-3} - 2 U' \left(|\phi_{,m}|^2 \right) \right) \, \left(\overline{\nabla\!\!\!\!\nabla\, \phi} \cdot_{TN} \partial_m \right) \cdot_{\phi^* T^* S} \phi^* g_S \otimes \partial_m \\
= \; & \phi_{,t} \cdot_{\phi^* T^* S} \phi^* g_S \otimes \partial_t + \left(\phi^* Q \; |\phi_{,m}|^{-3} - 2 U' \left(|\phi_{,m}|^2 \right) \right) \, \phi_{,m} \cdot_{\phi^* T^* S} \phi^* g_S \otimes \partial_m .
\end{aligned}
$$

The divergence of $(\nabla\!\!\!\!\nabla\, \phi)^* \, L_{,v}$ must be computed. Because the base manifold is $N = M \times I$, the divergence is computed by contracting the covariant derivative of $(\nabla\!\!\!\!\nabla\, \phi)^* \, L_{,v}$ with the identity tensor field on TN, which is $\partial_m \otimes dm + \partial_t \otimes dt$. Note that $\nabla_{\partial_m} dm = 0$ and $\nabla_{\partial_t} dt$ by construction of the coordinates m and t. The relevant quantities are

$$\begin{aligned}
&(\mathring{\nabla}\,\phi)^* \, L_{,v} \cdot_{T^*N} dm \\
&= \, (\phi_{,t} \cdot_{\phi^* T^* S} \phi^* g_S \otimes \partial_t \\
&\quad + \left(\phi^* Q \, |\phi_{,m}|^{-3} - 2U' \left(|\phi_{,m}|^2\right)\right) \, \phi_{,m} \cdot_{\phi^* T^* S} \phi^* g_S \otimes \partial_m \Big) \cdot_{T^*N} dm \\
&= \phi_{,t} \cdot_{\phi^* T^* S} \phi^* g_S \otimes \partial_t \cdot dm \\
&\quad + \left(\phi^* Q \, |\phi_{,m}|^{-3} - 2U' \left(|\phi_{,m}|^2\right)\right) \, \phi_{,m} \cdot_{\phi^* T^* S} \phi^* g_S \otimes \partial_m \cdot_{T^*N} dm \\
&= \, \left(\phi^* Q \, |\phi_{,m}|^{-3} - 2U' \left(|\phi_{,m}|^2\right)\right) \, \phi_{,m} \cdot_{\phi^* T^* S} \phi^* g_S,
\end{aligned}$$

$$\begin{aligned}
&(\mathring{\nabla}\,\phi)^* \, L_{,v} \cdot_{T^*N} dt \\
&= \, (\phi_{,t} \cdot_{\phi^* T^* S} \phi^* g_S \otimes \partial_t \\
&\quad + \left(\phi^* Q \, |\phi_{,m}|^{-3} - 2U' \left(|\phi_{,m}|^2\right)\right) \, \phi_{,m} \cdot_{\phi^* T^* S} \phi^* g_S \otimes \partial_m \Big) \cdot_{T^*N} dt \\
&= \phi_{,t} \cdot_{\phi^* T^* S} \phi^* g_S \otimes \partial_t \cdot dt \\
&\quad + \left(\phi^* Q \, |\phi_{,m}|^{-3} - 2U' \left(|\phi_{,m}|^2\right)\right) \, \phi_{,m} \cdot_{\phi^* T^* S} \phi^* g_S \otimes \partial_m \cdot_{T^*N} dt \\
&= \phi_{,t} \cdot_{\phi^* T^* S} \phi^* g_S,
\end{aligned}$$

$$\nabla \phi^* g_S = \phi^* \nabla g_S \cdot \mathring{\nabla}\,\phi = \phi^* 0 \cdot \mathring{\nabla}\,\phi = 0,$$

and

$$\begin{aligned}
&\operatorname{Div}_N \left((\mathring{\nabla}\,\phi)^* \, L_{,v}\right) \\
&= \nabla_{\partial_m} \left((\mathring{\nabla}\,\phi)^* \, L_{,v}\right) \cdot_{T^*N} dm + \nabla_{\partial_t} \left((\mathring{\nabla}\,\phi)^* \, L_{,v}\right) \cdot_{T^*N} dt \\
&= \nabla_{\partial_m} \left((\mathring{\nabla}\,\phi)^* \, L_{,v} \cdot_{T^*N} dm\right) + \nabla_{\partial_t} \left((\mathring{\nabla}\,\phi)^* \, L_{,v} \cdot_{T^*N} dt\right) \\
&= \nabla_{\partial_m} \left(\left(\phi^* Q \, |\phi_{,m}|^{-3} - 2U' \left(|\phi_{,m}|^2\right)\right) \, \phi_{,m} \cdot_{\phi^* T^* S} \phi^* g_S\right) + \nabla_{\partial_t} \left(\phi_{,t} \cdot_{\phi^* T^* S} \phi^* g_S\right) \\
&= \, \left[\nabla_{\partial_m} \left(\left(\phi^* Q \, |\phi_{,m}|^{-3} - 2U' \left(|\phi_{,m}|^2\right)\right) \, \phi_{,m}\right) + \nabla_{\partial_t} \phi_{,t}\right] \cdot_{\phi^* T^* S} \phi^* g_S.
\end{aligned}$$

It will simplify things to contract the whole Euler-Lagrange equation with $\phi^* g_S^{-1}$.

$$\begin{aligned}
\nabla_{\partial_m} |\phi_{,m}|^2 &= \nabla_{\partial_m} \left(\phi_{,m} \cdot_{\phi^*T^*S} \phi^* g_S \cdot_{\phi^*TS} \phi_{,m}\right) \\
&= \nabla_{\partial_m} \phi_{,m} \cdot_{\phi^*T^*S} \phi^* g_S \cdot_{\phi^*TS} \phi_{,m} + \phi_{,m} \cdot_{\phi^*T^*S} \phi^* g_S \cdot_{\phi^*TS} \nabla_{\partial_m} \phi_{,m} \\
&= 2\phi_{,m} \cdot_{\phi^*T^*S} \phi^* g_S \cdot_{\phi^*TS} \nabla_{\partial_m} \phi_{,m}, \\
\nabla_{\partial_m} |\phi_{,m}|^k &= \nabla_{\partial_m} \left(|\phi_{,m}|^2\right)^{k/2} \\
&= \frac{k}{2} \left(|\phi_{,m}|^2\right)^{k/2-1} \nabla_{\partial_m} |\phi_{,m}|^2 \\
&= \frac{k}{2} \left(|\phi_{,m}|^2\right)^{(k-2)/2} 2\phi_{,m} \cdot_{\phi^*T^*S} \phi^* g_S \cdot_{\phi^*TS} \nabla_{\partial_m} \phi_{,m} \\
&= k \, |\phi_{,m}|^{k-2} \, \phi_{,m} \cdot_{\phi^*T^*S} \phi^* g_S \cdot_{\phi^*TS} \nabla_{\partial_m} \phi_{,m},
\end{aligned}$$

$$\begin{aligned}
&\operatorname{Div}_N \left(\left(\nabla \phi\right)^* L_{,v}\right) \cdot_{\phi^*TS} \phi^* g_S^{-1} \\
&= \nabla_{\partial_m} \left(\left(\phi^* Q \, |\phi_{,m}|^{-3} - 2U'\left(|\phi_{,m}|^2\right)\right) \phi_{,m}\right) + \nabla_{\partial_t} \phi_{,t} \\
&= \nabla_{\partial_m} \left(\phi^* Q \, |\phi_{,m}|^{-3} - 2U'\left(|\phi_{,m}|^2\right)\right) \phi_{,m} \\
&\quad + \left(\phi^* Q \, |\phi_{,m}|^{-3} - 2U'\left(|\phi_{,m}|^2\right)\right) \nabla_{\partial_m} \phi_{,m} + \nabla_{\partial_t} \phi_{,t} \\
&= \left[|\phi_{,m}|^{-3} \, \nabla_{\partial_m} \phi^* Q + \phi^* Q \, \nabla_{\partial_m} |\phi_{,m}|^{-3} - 2U''\left(|\phi_{,m}|^2\right) \nabla_{\partial_m} |\phi_{,m}|^2\right] \phi_{,m} \\
&\quad + \left(\phi^* Q \, |\phi_{,m}|^{-3} - 2U'\left(|\phi_{,m}|^2\right)\right) \nabla_{\partial_m} \phi_{,m} + \nabla_{\partial_t} \phi_{,t} \\
&= \left[|\phi_{,m}|^{-3} \, \phi^* \nabla Q \cdot_{\phi^*TS} \phi_{,m}\right] \phi_{,m} \\
&\quad + \left[\phi^* Q \left(-3 \, |\phi_{,m}|^{-5} \, \phi_{,m} \cdot_{\phi^*T^*S} \phi^* g_S \cdot_{\phi^*TS} \nabla_{\partial_m} \phi_{,m}\right)\right] \phi_{,m} \\
&\quad - \left[2U''\left(|\phi_{,m}|^2\right) 2\phi_{,m} \cdot_{\phi^*T^*S} \phi^* g_S \cdot_{\phi^*TS} \nabla_{\partial_m} \phi_{,m}\right] \phi_{,m} \\
&\quad + \left(\phi^* Q \, |\phi_{,m}|^{-3} - 2U'\left(|\phi_{,m}|^2\right)\right) \nabla_{\partial_m} \phi_{,m} + \nabla_{\partial_t} \phi_{,t} \\
&= |\phi_{,m}|^{-3} \left(\phi^* \nabla Q \cdot_{\phi^*TS} \phi_{,m}\right) \phi_{,m} \\
&\quad - \left(3\phi^* Q \, |\phi_{,m}|^{-5} + 4U''\left(|\phi_{,m}|^2\right)\right) \left(\phi_{,m} \cdot_{\phi^*T^*S} \phi^* g_S \cdot_{\phi^*TS} \nabla_{\partial_m} \phi_{,m}\right) \phi_{,m} \\
&\quad + \left(\phi^* Q \, |\phi_{,m}|^{-3} - 2U'\left(|\phi_{,m}|^2\right)\right) \nabla_{\partial_m} \phi_{,m} + \nabla_{\partial_t} \phi_{,t}.
\end{aligned}$$

Note that

$$\begin{aligned}
\left(\nabla \phi\right)^* L_{,\sigma} \cdot_{\phi^*TS} \phi^* g_S^{-1} &= \left(- |\phi_{,m}|^{-1} \, \phi^* \nabla Q\right) \cdot_{\phi^*TS} \phi^* g_S^{-1} \\
&= - |\phi_{,m}|^{-1} \, \phi^* \operatorname{Grad} Q.
\end{aligned}$$

Finally, the Euler-Lagrange equation (contracted with $\phi^* g_S^{-1}$ can now be expressed.

$$\begin{aligned}
0 &= \left[\left(\nabla \phi \right)^* L_{,\sigma} - \operatorname{Div}_N \left(\left(\nabla \phi \right)^* L_{,v} \right) \right] \cdot_{\phi^* TS} \phi^* g_S^{-1} \\
&= \left(\nabla \phi \right)^* L_{,\sigma} \cdot_{\phi^* TS} \phi^* g_S^{-1} - \operatorname{Div}_N \left(\left(\nabla \phi \right)^* L_{,v} \right) \cdot_{\phi^* TS} \phi^* g_S^{-1} \\
&= - \left| \phi_{,m} \right|^{-1} \phi^* \operatorname{Grad} Q \\
&\quad - \left| \phi_{,m} \right|^{-3} \left(\phi^* \nabla Q \cdot_{\phi^* TS} \phi_{,m} \right) \phi_{,m} \\
&\quad + \left(3 \phi^* Q \left| \phi_{,m} \right|^{-5} + 4 U'' \left(\left| \phi_{,m} \right|^2 \right) \right) \left(\phi_{,m} \cdot_{\phi^* T^* S} \phi^* g_S \cdot_{\phi^* TS} \nabla_{\partial_m} \phi_{,m} \right) \phi_{,m} \\
&\quad - \left(\phi^* Q \left| \phi_{,m} \right|^{-3} - 2 U' \left(\left| \phi_{,m} \right|^2 \right) \right) \nabla_{\partial_m} \phi_{,m} + \nabla_{\partial_t} \phi_{,t}.
\end{aligned}$$

12.3.3 Formulas in Graph Coordinates

Let the manifold S be defined as the graph of the function $f \colon \mathbb{R}^{\dim S} \to \mathbb{R}$ embedded in the Euclidean space $\mathbb{R}^{\dim S + 1}$, inheriting its metric structure g_S from the embedding by asserting that the embedding is an isometry. If $X = \left(X^i \right)$ gives standard coordinates on $\mathbb{R}^{\dim S}$, then a coordinatized tangent vector $(X, V) \in TS$ is embedded in $\mathbb{R}^{\dim S + 1}$ as $\left(\left(X, f(X) \right), \left(V, Df(X) \cdot V \right) \right)$. Here, D indicates the total differential in the elementary vector calculus sense. The metric g_S is then given in coordinates by

$$\begin{aligned}
(X, U) \cdot g_S(X) \cdot (X, V) &:= \left(U, Df(X) \cdot U \right) \cdot \left(V, Df(X) \cdot V \right) \\
&= U \cdot V + U \cdot Df(X) \otimes Df(X) \cdot V \\
&= U \cdot \left(\mathbb{I} + Df(X) \otimes Df(X) \right) \cdot V,
\end{aligned}$$

showing that the coordinate expression for g_S is

$$g_S(X) = \mathbb{I} + Df(X) \otimes Df(X).$$

It is a straightforward calculation to compute the inverse of a tensor [field] of the form identity-plus-rank-one-tensor. In this case,

$$g_S^{-1}(X) = \mathbb{I} - \frac{1}{1 + \left| Df(X) \right|^2} Df(X) \otimes Df(X).$$

The Christoffel symbols of the linear covariant derivative on S are given in terms of $g_S(X)$ and $g_S^{-1}(X)$ and can be shown via another straightforward calculation. Letting Γ denote the Christoffel symbols,

$$\Gamma(X) = \frac{1}{1 + \left| Df \right|^2} Df \otimes D^2 f,$$

where the indices come in the order k, i, j.

12.3.4 A Particular Choice of Graph Manifold

If the manifold S is defined as the graph of the function $f(X) = -|X|^2$, then $Df(X) = -2X$ and $D^2 f(X) = -2\mathbb{I}$. In this case, the metric, inverse metric, and Christoffel symbols have expressions

$$\begin{aligned} g_S(X) &= \mathbb{I} + (-2X) \otimes (-2X) \\ &= \mathbb{I} + 4X \otimes X, \\ g_S^{-1}(X) &= \mathbb{I} - \frac{1}{1 + |-2X|^2} (-2X) \otimes (-2X) \\ &= \mathbb{I} - \frac{4}{1 + 4|X|^2} X \otimes X, \\ \Gamma(X) &= \frac{1}{1 + |-2X|^2} (-2X) \otimes (-2\mathbb{I}) \\ &= \frac{4}{1 + 4|X|^2} X \otimes \mathbb{I}. \end{aligned}$$

12.3.5 Christoffel Symbols for Pullback

Let $\Gamma(E)$ be the Christoffel symbols for linear covariant derivative ∇^E on a vector bundle $E \to N$, expressed with respect to the frame (e_i) (and local coordinates on N), and let $\psi \colon M \to N$. Then

$$\begin{aligned} \Gamma(\psi^* E)_{ij}^k \ \psi^* e_k &= \nabla_{\partial_i}^{\psi^* E} \psi^* e_j \\ &= \psi^* \nabla^E e_j \cdot \overline{\nabla \psi} \cdot \partial_i \\ &= \psi^* \nabla^E e_j \cdot \psi_{,i}^\ell \, \psi^* \partial_\ell \\ &= \psi_{,i}^\ell \psi^* \nabla_{\partial_\ell}^E e_j \\ &= \psi_{,i}^\ell \psi^* \left(\Gamma(E)_{\ell j}^k \, e_k \right) \\ &= \psi_{,i}^\ell \psi^* \Gamma(E)_{\ell j}^k \ \psi^* e_k \\ \implies \Gamma(\psi^* E)_{ij}^k &= \psi_{,i}^\ell \psi^* \Gamma(E)_{\ell j}^k . \end{aligned}$$

This could be phrased abstractly as

$$\Gamma(\psi^* E) = \Gamma(E) : (D\psi \boxtimes \mathbb{I}_E),$$

where $D\psi$ is the Jacobian matrix of ψ in coordinates, $\mathbb{I}_E \in \Gamma(E \otimes E^*)$ is the identity tensor on E, and $D\psi \boxtimes \mathbb{I}_E$ is their parallel tensor product.

In the special case where $E = TN$ and ∇^E is symmetric (e.g. the Levi-Civita covariant derivative induced by a Riemannian metric), the Christoffel symbols have a symmetry $\Gamma_{ij}^k = \Gamma_{ji}^k$, and the formula can be simplified to

$$\Gamma(\psi^* TN) = \Gamma(TN) \cdot D\psi.$$

12.3.6 Simplification of Euler-Lagrange Equation in the Radial Case

With $\phi(m) := \rho(\cos m, \sin m)$, $f(X) := -|X|^2$, and $Q := 2J\,f$, formulas for the relevant expressions in graph coordinates are as follows (see Section 12.3.3),

$$\begin{aligned}
\phi^* Q &= Q(\phi(m)) = 2J\,f(\phi(m)) = -2J\,|\rho(\cos m, \sin m)|^2 = -2J\rho^2, \\
\nabla f(X) &= -2X, \\
\nabla Q(X) &= -4JX, \\
\phi^* \nabla Q &= \nabla Q(\phi(m)) = -4J\rho(\cos m, \sin m), \\
g_S(X) &= \mathbb{I} + 4X \otimes X, \\
g_S^{-1}(X) &= \mathbb{I} - \frac{4}{1 + 4|X|^2} X \otimes X, \\
\operatorname{Grad} Q(X) &= \nabla Q(X) \cdot g_S^{-1}(X) = -4J\left(X - \frac{4|X|^2}{1 + 4|X|^2} X\right) = -\frac{4J}{1 + 4|X|^2} X, \\
(\phi^* \operatorname{Grad} Q)(m) &= \operatorname{Grad} Q(\phi(m)) \\
&= -\frac{4J}{1 + 4|\rho(\cos m, \sin m)|^2} \rho(\cos m, \sin m) \\
&= -\frac{4J\rho}{1 + 4\rho^2} (\cos m, \sin m), \\
\phi'(m) &= \rho(-\sin m, \cos m), \\
|\phi'|^2 &= \phi'(m) \cdot g_S(\phi(m)) \cdot \phi'(m) \\
&= \rho(-\sin m, \cos m) \cdot [\mathbb{I} + 4\phi(m) \otimes \phi(m)] \cdot \rho(-\sin m, \cos m) \\
&= \rho^2 |(-\sin m, \cos m)|^2 + 4\rho^4 (-\sin m, \cos m) \cdot (\cos m, \sin m) \\
&= \rho^2, \\
U'\left(|\phi'|^2\right) &= 1 - |\phi'|^{-4} = 1 - \rho^{-4}.
\end{aligned}$$

Note that because $|\phi'|^2$ is constant,

$$0 = \frac{d}{dm} |\phi'|^2 = \frac{d}{dm} \left(\phi' \cdot \phi^* g_S \cdot \phi'\right) = 2\nabla_{\partial_m} \phi' \cdot \phi^* g_S \cdot \phi'.$$

Additionally,

$$\phi^* \nabla Q \cdot \phi' = -4J\rho\,(\cos m, \sin m) \cdot \rho\,(-\sin m, \cos m) = 0.$$

Finally, in graph coordinates,

$$\begin{aligned}
\mathbb{I} : \left(\phi' \otimes \phi'\right) &= \mathbb{I} : \left(\rho\,(-\sin m, \cos m) \otimes \rho\,(-\sin m, \cos m)\right) \\
&= \rho^2\,(-\sin m, \cos m) \cdot (-\sin m, \cos m) \\
&= \rho^2, \\
\nabla_{\partial_m} \phi' &= \phi'' + \Gamma\,(\phi\,(m))\left(\phi' \otimes \phi'\right) \\
&= -\,\rho\,(\cos m, \sin m) \\
&\quad + \frac{4}{1 + 4\,|\rho\,(\cos m, \sin m)|^2}\,\rho\,(\cos m, \sin m) \otimes \mathbb{I} : \left(\phi' \otimes \phi'\right) \\
&= \left[-1 + \frac{4\rho^2}{1 + 4\rho^2}\right]\rho\,(\cos m, \sin m) \\
&= -\,\frac{\rho}{1 + 4\rho^2}\,(\cos m, \sin m)\,.
\end{aligned}$$

Two of the terms in the Euler-Lagrange equation drop out and it becomes

$$\begin{aligned}
0 &= -\left|\phi'\right|^{-1}\,\phi^*\,\mathrm{Grad}\,Q - \left(\phi^* Q\,\left|\phi'\right|^{-3} - 2U'\left(\left|\phi'\right|^2\right)\right)\nabla_{\partial_m}\phi' \\
&= -\,\rho^{-1}\left(-\frac{4J\rho}{1 + 4\rho^2}\,(\cos m, \sin m)\right) \\
&\quad - \left(-2J\rho^2\,\rho^{-3} - 2\left(1 - \rho^{-4}\right)\right)\left(-\frac{\rho}{1 + 4\rho^2}\,(\cos m, \sin m)\right) \\
&= \left[\frac{4J}{1 + 4\rho^2} - 2\left(J\rho^{-1} + 1 - \rho^{-4}\right)\left(\frac{\rho}{1 + 4\rho^2}\right)\right](\cos m, \sin m)\,.
\end{aligned}$$

12.3.7 Simplification of Dynamic Euler-Lagrange Equation in the Radial Case

With $\phi\,(m,t) := \rho\,(t)\,(\cos m, \sin m)$, $f\,(X) := -\,|X|^2$, and $Q := 2J\,f$, formulas for the relevant expressions in graph coordinates are as follows (see Section 12.3.3),

The following are some calculations necessary for simplifying the Euler-Lagrange equation in this radially symmetric case.

$$\begin{aligned}
\phi_{,m} &= \rho(t)(-\sin m, \cos m), \\
|\phi_{,m}|^2 &= \rho(t)^2, \\
\phi^* Q &= -2J\rho(t)^2 \\
\phi^* \operatorname{Grad} Q &= -\frac{4J\rho(t)}{1+4\rho(t)^2}(\cos m, \sin m), \\
\phi^* \nabla Q \cdot \phi_{,m} &= 0, \\
\phi_{,m} \cdot \phi^* g_S \cdot \nabla_{\partial_m} \phi_{,m} &= 0, \\
\nabla_{\partial_m} \phi_{,m} &= -\frac{\rho(t)}{1+4\rho(t)^2}(\cos m, \sin m), \\
\phi_{,t} &= \rho'(t)(\cos m, \sin m), \\
\mathbb{I} : (\phi_{,t} \otimes \phi_{,t}) &= \mathbb{I} : \left(\rho'(t)(\cos m, \sin m) \otimes \rho'(t)(\cos m, \sin m)\right) \\
&= \rho'(t)^2 (\cos m, \sin m) \cdot (\cos m, \sin m) \\
&= \rho'(t)^2, \\
\nabla_{\partial_t} \phi_{,t} &= \phi_{,tt} + \Gamma(\phi(m)) : (\phi_{,t} \otimes \phi_{,t}) \\
&= \rho''(t)(\cos m, \sin m) \\
&+ \frac{4}{1+4|\rho(t)(\cos m, \sin m)|^2} \rho(t)(\cos m, \sin m) \otimes \mathbb{I} : (\phi_{,t} \otimes \phi_{,t}) \\
&= \left[\rho''(t) + \frac{4\rho(t)\rho'(t)^2}{1+4\rho(t)^2}\right](\cos m, \sin m).
\end{aligned}$$

The Euler-Lagrange equation becomes

$$\begin{aligned}
0 = &-\rho(t)^{-1}\left(-\frac{4J\rho(t)}{1+4\rho(t)^2}(\cos m, \sin m)\right) \\
&-\left(-2J\rho(t)^2\,\rho(t)^{-3} - 2\left(1-\rho(t)^{-4}\right)\right)\left(-\frac{\rho(t)}{1+4\rho(t)^2}(\cos m, \sin m)\right) \\
&+\left[\rho''(t) + \frac{4\rho(t)\rho'(t)^2}{1+4\rho(t)^2}\right] \\
= &\left[\rho''(t) + \frac{4\rho(t)\rho'(t)^2}{1+4\rho(t)^2} - 2\rho(t)^{-3}\,\frac{\rho(t)^4 - J\rho(t)^3 - 1}{1+4\rho(t)^2}\right](\cos m, \sin m).
\end{aligned}$$

Part III

Strongly Typed Tensor Calculus Formalism

Many important differential equations have a variational origin, being derived as the Euler-Lagrange equations for a particular functional on some space of functions. The variational approach lends itself particularly to physics, in which conservation of energy or minimization of action is a central concept. The naturality of such formulations can not be understated, as solutions to such problems often depend critically on the inherent geometry of the underlying objects. For example, solutions to Laplace's equation for a real valued function (e.g. modeling steady-state heat flow) on a Riemannian manifold depend qualitatively on the topology of the manifold (e.g. harmonic functions on a closed Riemannian manifold are necessarily constant, which makes sense geometrically because there is no boundary through which heat can escape).

A central concept in the field of software design is that of **information hiding** (see [Par72]), in which a computer program is organized into modules, each presenting an abstract public interface. Other parts of the program can interact only through the presented interface, and the details of how each module works are hidden, thereby preventing interference in the implementation details which are not required by the inherent structure of the module. This concept has clear usefulness in the field of mathematics as well. For example, there are several formulations of the real numbers (e.g. equivalence classes of Cauchy sequences of rational numbers, Dedekind cuts, decimal expansions, etc), but their particulars are instances of what are known as **implementation details**, and the details of each particular implementation are irrelevant in most areas of mathematics, which only use the inherent properties of the real numbers as a complete, totally ordered field. Of course, at certain levels, it is useful or necessary to "open up the box" [go past the public interface] and work with a particular representation of the real numbers.

Information hiding is characteristic of abstract mathematics, in which general results are proved about abstract mathematical objects without using any particular implementation of said objects. These results can then be used modularly in other proofs, just as the functionality of a computer program is organized into modularized objects and functions. For example, a fixed point theorem for contractive mappings on closed sets in Banach spaces, but a particular application of this theorem renders an existence and uniqueness theorem for first order ODEs (see [Wal98, pgs. 59, 62]).

A loose conceptual analogy for modularity is that of diagonalizing a linear operator. A basis of eigenvectors are chosen so that the action of the linear operator on each eigenspace has a particularly simple expression, and distinct eigenspaces do not interact with respect to the operator's action. In this analogy, the eigenvectors then correspond to individual lemmas, and the linear operator corresponds to a large theorem which uses each lemma. Decomposing the proof of the main result in terms of non-interacting lemmas simplifies the proof considerably, just as it simplifies the quantification of the linear operator. The term "orthogonal" has been borrowed by software design to describe two program modules whose functionality is independent (see [Ray03, Chapter 4, Section 2]). Orthogonality in software design is highly desirable as it generally eases program implementation and program correctness verification, as the human designers are only capable of keeping track of a certain finite number of details simultaneously (see [Mil55]). The scope of each detail level of the design is limited in complexity, making the overall design easier to comprehend.

This technique in software design carries over directly to proof design, where it is desirable (elegant) to write proofs and do calculations without introducing extraneous details, such as choice of bases in vector spaces or local coordinates in manifolds. Because such choices are generally non-unique, they can often obscure the inherent structure of the relevant objects by introducing artifacts arising from properties of the particular details used to implement said objects. For example, the choice of a particular local coordinate chart on a manifold artificially imposes an additive structure on a neighborhood of the manifold, but such a structure has nothing to do with the inherent geometry of the manifold. Furthermore, the descent to this "lower level" of calculation discards some type information, representing points in a manifold as Euclidean vectors, thereby losing the ability to distinguish points from different manifolds, or even different localities in the same manifold.

This paper makes a particular emphasis on natural formulations and calculations in order to expose the underlying geometric structures rather than relying on coordinate-based expressions. The construction of the "full" direct sum and "full" tensor product bundles are used in combination with induced covariant derivatives to this end.

13 Notation and Conventions

Let all vector spaces, manifolds and [fiber] bundles be real and finite-dimensional unless otherwise noted (this allows the canonical identification $V^{**} \cong V$ for a vector space or vector bundle V), and let all tensor products be over $\mathbb{R}$. The unqualified term "bundle" will mean "fiber bundle". The Einstein summation convention will be assumed in situations when indexed tensors are used for computation.

Unary operators are understood to have higher binding precedence than binary operators, and super and subscripts are understood to have the highest binding precedence. For example, the expression $\nabla X_{,M} \circ \phi$ would be parenthesized as $(\nabla (X_{,M})) \circ \phi$.

Apart from the obvious purpose of providing a concise and central reference for the notation in this paper, the following notation index serves to illustrate the use of telescoping notation (see Section 15). The high-level (terse notation which requires the reader to do more work in type inference but is more agile), mid-level, and low-level (completely type-specified, requiring little work on the part of the reader) notations are presented side-by-side with their definitions.

Let $I \subseteq \mathbb{R}$ be a neighborhood of 0, let ϵ, i each be coordinates on I, let $A, A_1, \dots, A_n, B$ be sets, let M, N be manifolds, let $\phi \in C^\infty(M, N)$, let $\pi_M^A \colon A \to M$ and $\pi_N^H \colon H \to N$ and be vector bundles, where $A = E, F, F_1, \dots, F_n, G$, let $U, V, V_1, \dots, V_n, W$ be vector spaces, and let $c_i \in \Gamma(F_i \otimes_M T^*M)$ such that

$$c_1 \oplus_M \cdots \oplus_M c_n \in \Gamma((F_1 \oplus_M \cdots \oplus_M F_n) \otimes_M T^*M)$$

is a vector bundle isomorphism.

High-	Mid-	Low-level	Description
Variations; variational derivatives; tangent vectors.			
m_ϵ		m	Variation of a point in M; $I \ni \epsilon \mapsto m_\epsilon \in M$; $m\colon I \to M$.
δ		δ_ϵ	Variational derivative; $\delta_\epsilon := \frac{\partial}{\partial \epsilon} \|_{\epsilon=0}$.
δm_ϵ		$\delta_\epsilon m$	Tangent vector; linearization of a variation; $\delta m_\epsilon \in T_{m_0} M$; $\delta_\epsilon m \in T_{m(0)} M$.
Projection maps; canonical isomorphisms; bundle-related maps and spaces.			
Pr	Pr_i	$\mathrm{Pr}_i^{A_1 \times \cdots \times A_n}$	Set-theoretic projection onto ith factor or named factor;
	Pr_{A_i}	$\mathrm{Pr}_{A_i}^{A_1 \times \cdots \times A_n}$	$\mathrm{Pr}_i^{A_1 \times \cdots \times A_n}\colon A_1 \times \cdots \times A_n \to A_i$.
ι	ι_B, ι^A	ι_B^A	Canonical isomorphism; $\iota_B^A\colon A \to B$; $\iota_A^B := \left(\iota_B^A\right)^{-1}$.
π	π_M, π^F	π_M^F	Bundle projection map; $\pi_M^F\colon F \to M$.
ρ	ρ_H, $\rho^{\phi^* H}$	$\rho_H^{\phi^* H}$	Pullback bundle fiber projection map; $\rho_H^{\phi^* H}\colon \phi^* H \to H$.
Trivial bundle constructions and projection maps.			
$M \times N \to N$		$M \mathbin{\times\!\!\!\!\to} N \to N$	Trivial bundle over N; $M \mathbin{\times\!\!\!\!\to} N := M \times N$; $\pi_N^{M \mathbin{\times\!\!\!\!\to} N}\colon M \mathbin{\times\!\!\!\!\to} N \to N$, $(m,n) \mapsto n$.
$M \times N \to M$		$M \mathbin{\gets\!\!\!\!\times} N \to M$	Trivial bundle over M; $M \mathbin{\gets\!\!\!\!\times} N := M \times N$; $\pi_M^{M \mathbin{\gets\!\!\!\!\times} N}\colon M \mathbin{\gets\!\!\!\!\times} N \to M$, $(m,n) \mapsto m$.
Shared base-space bundle constructions and projection maps.			
$E \times F \to M$		$E \times_M F \to M$	Direct product; $E \times_M F := \coprod_{m \in M} E_m \times F_m$; $\pi_M^{E \times_M F}(e,f) := \pi_M^E(e) \equiv \pi_M^F(f)$.
$E \oplus F \to M$		$E \oplus_M F \to M$	Whitney sum; $E \oplus_M F := \coprod_{m \in M} E_m \oplus F_m$; $\pi_M^{E \oplus_M F}(e \oplus f) := \pi_M^E(e) \equiv \pi_M^F(f)$.
$E \otimes F \to M$		$E \otimes_M F \to M$	Tensor product; $E \otimes_M F := \coprod_{m \in M} E_m \otimes F_m$; $\pi_M^{E \otimes_M F}\left(c^{ij} e_i \otimes f_j\right) := \pi_M^E(e_k) \equiv \pi_M^F(f_\ell)$ (for any k, ℓ).
Separate base-space bundle constructions and projection maps.			
$E \times H \to M \times N$		$E \times_{M \times N} H \to M \times N$	Direct product; $E \times_{M \times N} H := \coprod_{(m,n) \in M \times N} E_m \times H_n$. $\pi_{M \times N}^{E \times_{M \times N} H}(e,h) := \left(\pi_M^E(e), \pi_N^H(h)\right)$.
$E \oplus H \to M \times N$		$E \oplus_{M \times N} H \to M \times N$	Whitney sum; $E \oplus_{M \times N} H := \coprod_{(m,n) \in M \times N} E_m \oplus H_n$. $\pi_{M \times N}^{E \oplus_{M \times N} H}(e \oplus h) := \left(\pi_M^E(e), \pi_N^H(h)\right)$.
$E \otimes H \to M \times N$		$E \otimes_{M \times N} H \to M \times N$	Tensor product; $E \otimes_{M \times N} H := \coprod_{(m,n) \in M \times N} E_m \otimes H_n$. $\pi_{M \times N}^{E \otimes_{M \times N} H}\left(c^{ij} e_i \otimes h_j\right) := \left(\pi_M^E(e_k), \pi_N^H(h_\ell)\right)$ (for any k, ℓ).

High-	Mid-	Low-level	Description
Trace; natural pairing; tensor/tensor field contraction. Simple tensor expressions are extended linearly.			
Tr		Tr_V	Trace on V; $\mathrm{Tr}_V : V^* \otimes V \to \mathbb{R}$, $\alpha \otimes v \mapsto \alpha(v)$.
$\alpha \cdot v$		$\alpha \cdot_V v$	Natural pairing; $\cdot_V : V^* \times V \to \mathbb{R}$, $(\alpha, v) \mapsto \alpha(v)$.
$A \cdot B$		$A \cdot_V B$	Tensor contraction; $\cdot_V : (U \otimes V^*) \times (V \otimes W) \to U \otimes W$,
			$(u \otimes \alpha) \cdot_V (v \otimes w) := u \otimes (\alpha \cdot_V v) \otimes w \equiv \alpha(v)\, u \otimes w$.
$S \cdot^n T$		$S \cdot_{V_1 \otimes \cdots \otimes V_n} T$	Alternate for $\cdot_V$, where $V = V_1 \otimes \cdots \otimes V_n$.
$S : T$, $S \cdot^2 T$		$S \cdot_{V_1 \otimes V_2} T$	Special notation for $n = 2$.
Tr		Tr_F	Trace on $F \to M$; $\mathrm{Tr}_F : \Gamma(F^* \otimes_M F) \to C^\infty(M, \mathbb{R})$,
			$[\mathrm{Tr}_F(\sigma \otimes_M f)](m) := \sigma(m) \cdot_{F_m} f(m)$ for $m \in M$.
$\sigma \cdot f$		$\sigma \cdot_F f$	Natural pairing; $\cdot_F : \Gamma(F^*) \times \Gamma(F) \to C^\infty(M, \mathbb{R})$,
			$(\sigma \cdot_F f)(m) := \sigma(m) \cdot_{F_m} f(m)$ for $m \in M$.
$A \cdot f$		$A \cdot_F f$	Natural pairing; $\cdot_F : \Gamma(E \otimes_M F^*) \times F \to E$,
			$(e \otimes_M \sigma) \cdot_F f := e(m)\, (\sigma(m) \cdot_{F_m} f) \in E_m$; $m := \pi_M^F(f)$.
$S \cdot T$		$S \cdot_F T$	Tensor field contraction; pointwise tensor contraction;
			$\cdot_F : \Gamma(E \otimes_M F^*) \times \Gamma(F \otimes_M G) \to \Gamma(E \otimes_M G)$,
			$[(e \otimes \sigma) \cdot_F (f \otimes g)](m) := (\sigma(m) \cdot_{F_m} f(m))\, e(m) \otimes g(m)$.
$S \cdot^n T$		$S \cdot_{F_1 \otimes_M \cdots \otimes_M F_n} T$	Alternate for $\cdot_F$, where $F = F_1 \otimes_M \cdots \otimes_M F_n$.
$S : T$, $S \cdot^2 T$		$S \cdot_{F_1 \otimes_M F_2} T$	Special notation for $n = 2$.
Permutations of tensors and tensor fields.			
A^σ, $A \cdot^n \sigma$		$A \cdot_{V_1^* \otimes \cdots \otimes V_n^*} \sigma$	Right-action of permutations on n-tensors/n-tensor fields;
			$(v_1 \otimes \cdots \otimes v_n)^\sigma := v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$; $(A^\sigma)^\tau = A^{\sigma\tau}$.
Spaces of sections of bundles.			
$\Gamma(H)$, $\Gamma(\pi_N^H)$			Space of smooth sections of the bundle π_N^H;
			$\Gamma(H) := \{h \in C^\infty(N, H) \mid \pi_N^H \circ h = \mathrm{Id}_N\}$.
$\Gamma_\phi(H)$, $\Gamma_\phi(\pi_N^H)$			Space of smooth sections of π_N^H along ϕ;
			$\Gamma_\phi(H) := \{h \in C^\infty(M, H) \mid \pi_N^H \circ h = \phi\}$.
Vertical bundle, pullback bundle, projection maps, pullback of sections.			
$VE \to E$			Vertical bundle over $E \to M$; $VE := \ker T\pi_M^E \leq TE$.
			projection map $\pi_E^{VE} := \pi_E^{TE} \|_{VE}$.
$\phi^* H \to M$			Pullback bundle; $\phi^* H := \{(m, h) \in M \times H \mid \phi(m) = \pi(h)\}$.
			$\pi_M^{\phi^* H}(m, h) := m$; $\rho_H^{\phi^* H}(m, h) := h$.
$\phi^* h$			Pullback of section $h \in \Gamma(H)$; $\phi^* h \in \Gamma(\phi^* H)$
			defined by $\rho_H^{\phi^* H} \circ \phi^* h = h$; $h \in \Gamma(H)$.

High-	Mid-	Low-level	Description
Covariant derivatives; partial covariant derivatives.			
∇L	$\nabla\ L$ $\nabla^{M\to\mathbb{R}}L$	$\nabla^{\ M\to\mathbb{R}}L$	Natural linear covariant derivative; differential of functions; $\nabla^{\ M\to\mathbb{R}}L := dL \in \Gamma\left(T^*M\right)$, where $L \in C^\infty\left(M,\mathbb{R}\right)$.
∇X	$\nabla\ X$ $\nabla^{E}X$	$\nabla^{\ E}X$	Linear covariant derivative on vector bundle $E \to M$; $\nabla^{E}X \in \Gamma\left(E \otimes_M T^*M\right)$, where $X \in \Gamma\left(E\right)$.
$\nabla\phi$	$\boldsymbol{\nabla}\ \phi$ $\nabla^{M\to N}\phi$	$\boldsymbol{\nabla}^{\ M\to N}\phi$	Tangent map as tensor field; $\boldsymbol{\nabla}^{\ M\to N}\phi \in \Gamma\left(\phi^*TN \otimes_M T^*M\right)$, where $\phi \in C^\infty\left(M,N\right)$.
$\nabla\sigma$	$\nabla^{\phi}\sigma$	$\nabla^{\phi^*H}\sigma$	Pullback covariant derivative; $\sigma \in \Gamma\left(\phi^*H\right)$; defined by $\nabla^{\phi^*H}\phi^*h = \phi^*\nabla^H h \cdot_{\phi^*TN} \boldsymbol{\nabla}^{\ M\to N}\phi$; $h \in \Gamma\left(H\right)$.
$L_{,c_1},\ldots,L_{,c_n}$			Partial differential of functions; $L_{,c_i} \in \Gamma\left(F_i^*\right)$, defined by $\nabla^{\ M\to\mathbb{R}}L = \sum_{i=1}^n L_{,c_i} \cdot_{F_i} c_i$.
$X_{,c_1},\ldots,X_{,c_n}$			Partial linear covariant derivative; $X_{,c_i} \in \Gamma\left(E \otimes_M F_i^*\right)$, defined by $\nabla^{\ E}X = \sum_{i=1}^n X_{,c_i} \cdot_{F_i} c_i$.
$\phi_{,M_1},\ldots,\phi_{,M_n}$			Partial derivative decomposition of tangent map; $\phi_{,M_i} \in \Gamma\left(\phi^*TN \otimes_M \mathrm{Pr}_i^*\, T^*M_i\right)$, where $M = M_1 \times \cdots \times M_n$, $\mathrm{Pr}_i := \mathrm{Pr}_i^M$, and $\boldsymbol{\nabla}^{\ M\to N}\phi = \sum_{i=1}^n \phi_{,M_i} \cdot_{\mathrm{Pr}_i^*\,TM_i} \boldsymbol{\nabla}^{\ M\to M_i}\,\mathrm{Pr}_i$.
Covariant Hessians.			
$\nabla^2 L$ $\nabla\ \nabla\ L$	$\nabla^{T^*M}\nabla^{M\to\mathbb{R}}L$ $\nabla^{\ T^*M}\ \nabla^{\ M\to\mathbb{R}}L$		Covariant Hessian of functions; $\nabla^2 L \in \Gamma\left(T^*M \otimes T^*M\right)$; $L \in C^\infty\left(M,\mathbb{R}\right)$.
$\nabla^2 X$ $\nabla\ \nabla\ X$	$\nabla^{E\otimes T^*M}\nabla^{E}X$ $\nabla^{\ E\otimes_M T^*M}\ \nabla^{\ E}X$		Covariant Hessian on vector bundle $E \to M$; $\nabla^2 X \in \Gamma\left(E \otimes T^*M \otimes T^*M\right)$; $X \in \Gamma\left(E\right)$.
$\nabla^2\phi$ $\nabla\ \boldsymbol{\nabla}\ \phi$	$\nabla^{\phi^*TN\otimes T^*M}\nabla^{M\to N}\phi$ $\nabla^{\ \phi^*TN\otimes_M T^*M}\ \boldsymbol{\nabla}^{\ M\to N}\phi$		Covariant Hessian of maps; $\nabla^2\phi \in \Gamma\left(\phi^*TN \otimes T^*M \otimes T^*M\right)$. $\phi \in C^\infty\left(M,N\right)$.
Derivative conventions.			
$\nabla_X e$			Directional derivative notation; $\nabla_X e := \nabla e \cdot_{TM} X$.
$\nabla^n e \cdot^n \left(X_1 \otimes \cdots \otimes X_{n-1} \otimes X_n\right)$			Iterated covariant derivative convention; defined by $\left(\nabla_{X_n}\nabla^{n-1}e\right) \cdot^{n-1} \left(X_1 \otimes \cdots \otimes X_{n-1}\right)$.
$R\left(X,Y\right) := -\nabla_X\nabla_Y + \nabla_Y\nabla_X + \nabla_{[X,Y]}$			Curvature operator; $R\left(X,Y\right)e = \nabla^2 e : \left(X \otimes Y - Y \otimes X\right)$.
$z\colon M \to M \times I$, $m \mapsto (m,0)$			Evaluation-at-zero map.
$z^*\partial_i = \delta_i$			Pullback formulation of derivative-at-zero.

For more on relevant introductory theory on manifolds, bundles and Riemannian geometry, see [Lee06], [Lee09], [Mic08], [Lee97].

14 Using Strong Typing to Error-Check Calculations

Linear algebra is an excellent setting for discussion of the **strong typing** (see [Car93]) of a language, a concept used in the design of computer programming languages. The idea is that when the human-readable source code of a program is compiled (translated into machine-readable instructions), the compiler (the program which per-forms this translation) or runtime (the software which executes the code) verifies that the program objects are being used in a well-defined way, producing an error for each operation that is not well-defined. For example, a vector-type value would not be allowed to be added to a permutation-type value, even though tuples of unsigned integers (i.e. bytes) are used by the computer to represent both, and the computer's processing unit could add together their byte-valued representations. However, such an operation would be meaningless with respect to the types of the operands. The result of the operation would depend on the non-canonical choice of representation for each object. Strong type checking has the advantage of catching many programming errors, including most importantly those resulting from an inherent misuse of the program's objects. Within this paper, certain type-explicit notations will be used to provide forms of type awareness conducive to error-checking.

An important example of semi-strong typing in math is Penrose's abstract index notation (see [Pen04]), modeled on Einstein's summation convention, in which linear algebra and tensor calculus are implemented using indexed objects (tensors) having a certain number and order of "up" and "down" indices (an abstraction of the genuine basis/coordinate expressions in which the indexed objects are arrays of scalars/functions). A non-indexed tensor is a scalar value, a tensor having a single up or down index is a vector or covector value respectively, a tensor having an up and a down index is an endomorphism, and so forth. The tensors are contracted by pairing a certain number of up indices with the same number of down indices, resulting in an object having as indices the uncontracted indices.

For example, given a finite-dimensional inner product space (V, g), where g is a $\binom{0}{2}$-tensor (having the form g_{ij}, i.e. two down indices), a vector $v \in V$ is a $\binom{1}{0}$-tensor, and the length of v is $\sqrt{v^i g_{ij} v^j}$. If $\dim V > 1$, then $\bigwedge^2 V$ has positive dimension, its vectors each being $\binom{2}{0}$-tensors, and $G_{ijk\ell} := g_{ik} g_{j\ell} - g_{i\ell} g_{jk}$ is an inner product on $\bigwedge^2 V$ (which must be a $\binom{0}{4}$-tensor in order to contract with two $\binom{2}{0}$-tensors).

Certain type errors are detected by use of abstract index notation in the form of index mismatch. For example, with (V, g) as above, if $\alpha \in V^*$, then α is a $\binom{0}{1}$-tensor. Because of the repeated j down indices, the expression $g_{ij} \alpha_j$ typically indicates a type error; g_{ij} can't contract with α_j because of incompatible valence (valence being the number of up and down indices). Furthermore, multiplying a $\binom{0}{2}$-tensor with a $\binom{0}{1}$-

tensor without contraction should result in a $\binom{0}{3}$-tensor, which should be denoted using three indices, as in $g_{ij}\alpha_k$.

The only explicit type information provided by abstract index notation is that of valence. The "semi" qualifier mentioned earlier is earned by the lack of distinction between the different spaces in which the tensors reside. For example, if U, V, W are finite-dimensional vector spaces, then linear maps $A\colon U \to V$ and $B\colon V \to W$ can be written as $\binom{1}{1}$-tensors, and their composition $B \circ A\colon U \to W$ is written as the tensor contraction $(B \circ A)^i_j = B^i_k A^k_j$. However, while the expression $A^i_k B^k_j$ makes sense in terms of valence compatibility (i.e. grammatically), the composition "$A \circ B$" that it should represent is not well-defined. Thus this form of type error is not caught by abstract index notation, since the domains/codomains of the linear maps must be checked separately.

The use of dimensional analysis (the abstract use of units such as kilograms, seconds, etc) in Physics is an important precedent of strong typing. Each quantity has an associated "dimension" (this is a different meaning from the "dimension" of linear algebra) which is expressed as a fraction of powers of formal symbols. The ordinary algebraic rules for fractions and formal symbols are used for the dimensional components, with the further requirement that addition and equality may only occur between quantities having the same dimension.

For example, if E, M and C represent the dimensions of energy, mass and cost, respectively, and if the energy storage density $\rho\,\mathrm{E}/\mathrm{M}$ of a battery manufacturing process is known (having dimensions energy per mass) and the manufacturing weight yield $w\,\mathrm{M}/\mathrm{C}$ of the battery is known (having dimensions mass per cost), then under the algebraic conventions of dimensional analysis, calculating the energy storage per cost (which should have dimensions energy per cost) is simple;

$$\left(\rho\,\frac{\mathrm{E}}{\mathrm{M}}\right)\left(w\,\frac{\mathrm{M}}{\mathrm{C}}\right) = \rho w\,\frac{\mathrm{EM}}{\mathrm{MC}} = \rho w\,\frac{\mathrm{E}}{\mathrm{C}}$$

(the M symbols cancel in the fraction). Here, both ρ and w are real numbers, and besides using the well-definedness of real multiplication, no type-checking is done in the expression ρw.

A contrasting example is the quantity ρ/w, having dimensions EC/M^2. However, these dimensions may be considered to be meaningless in the given context. The quantity's type adds meaning to the real-valued quantity, and while the quantity is well-defined as a real number, the uselessness of the type may indicate that an error has been made in the calculations. For example, a type mismatch between the two sides of an equation is a strong indication of error.

This is also a convenient way to think about the chain rule of calculus. If $z\left(y\right)$, $y\left(x\right)$, and x measure real-valued quantities, then $z\left(y\left(x\right)\right)$ measures the quantity z with respect to quantity x. Using Z, Y and X for the dimensions of the quantities z, y and x respectively, the derivative $\frac{dz}{dx}$ has units Z/X. When worked out, the dimensions for the quantities on either side of the equation $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$ will match exactly, having a non-coincidental similarity to the calculation in the battery product example.

15 Telescoping Notation (aka Don't Fear the Verbosity)

Many of the computations developed in this paper will appear to be overly pedantic, owing to the decoration-heavy notation that will be introduced in Section 16. This decoration is largely for the purpose of tracking the myriad of types in the type system and to assist the human reader or writer in making sense of and error-checking the expressions involved. The pedantry in this paper plays the role of introducing the technique. The notation is designed to **telescope**[18], meaning that there is a spectrum of notational decoration; from

- pedantically type-specified, verbose, and decoration-heavy, where [almost] no types must be inferred from context and there is little work or expertise required on the part of the reader, to
- somewhat decorated but more compact, where the reader must do a little bit of thinking to infer some types, all the way to
- tersely notated with minimal type decoration, where [almost] all types must be inferred from context and the reader must either do a lot of thinking or be relatively experienced.

Additionally, some of the chosen symbols are meant to obey the same telescoping range of specificity. For example, compare n-fold tensor contraction $\cdot^n$ with type-specified $\cdot_{V_1 \otimes \cdots \otimes V_n}$ as discussed in Section 16, or the symbols ∇, $\mathring{\nabla}$, and $\overline{\nabla}$ as discussed in

[18] Credit for the notion of telescoping notation is due in part to David DeConde, during one of many enjoyable and insightful conversations.

Section 23. Tersely notated computations can be seen in Section 23, while fully-verbose computations abound in the careful exposition of Part IV.

16 Strongly-Typed Linear Algebra via Tensor Products

A fully strongly typed formulation of linear algebra will now be developed which enjoys a level of abstraction and flexibility similar to that of Penrose's abstract index notation. Emphasis will be placed on notational and conceptual regularity via a tensor formalism, coupled with a notion of "untangled" expression which exploits and notationally depicts the associativity of linear composition.

If V denotes a finite-dimensional vector space, then let

$$\cdot_V : V^* \times V \to \mathbb{R},\ (\alpha, v) \mapsto \alpha(v)$$

denote the **natural pairing** on V, and denote $\cdot_V(\alpha, v)$ using the infix notation $\alpha \cdot_V v$. The natural pairing is a nondegenerate bilinear form and its bilinearity gives the expression $\alpha \cdot_V v$ multiplicative semantics (distributivity and commutativity with scalar multiplication), thereby justifying the use of the infix $\cdot$ operator normally reserved for multiplication. The natural pairing subscript V is seemingly pedantic, but will prove to be an invaluable tool for articulating and navigating the rich type system of the linear algebraic and vector bundle constructions used in this paper. When clear from context, the subscript V may be omitted.

Because V is finite-dimensional, it is reflexive (i.e. the canonical injection $V \to V^{**}$, $v \mapsto (\alpha \mapsto \alpha(v))$ is a linear isomorphism). Thus the natural pairing $\cdot_{V^*}$ on V^* can be written naturally as

$$\cdot_{V^*} : V \times V^* \to \mathbb{R},\ (v, \alpha) \mapsto \alpha(v).$$

Note that $\alpha \cdot_V v = v \cdot_{V^*} \alpha$. Though subtle, the distinction between $\cdot_V$ and $\cdot_{V^*}$ is important within the type system used in this paper.

Through a universal mapping property of multilinear maps, the bilinear forms $\cdot_V$ and $\cdot_{V^*}$ descend to the **natural trace** maps

$$\begin{aligned} \operatorname{Tr}_V &: V^* \otimes V \to \mathbb{R},\ \alpha \otimes v \mapsto \alpha(v), \text{ and} \\ \operatorname{Tr}_{V^*} &: V \otimes V^* \to \mathbb{R},\ v \otimes \alpha \mapsto \alpha(v), \end{aligned}$$

each extended linearly to non-simple tensors. These operations can also be called **tensor contraction**. Noting that $(V^* \otimes V)^*$ and $(V \otimes V^*)^*$ are canonically isomorphic to $V \otimes V^*$ and $V^* \otimes V$ respectively, then for each $A \in V^* \otimes V$ and $B \in V \otimes V^*$, it follows that $\operatorname{Tr}_V(A) = \operatorname{Id}_{V^*} \cdot_{V^* \otimes V} A$ and $\operatorname{Tr}_{V^*}(B) = \operatorname{Id}_V \cdot_{V \otimes V^*} B$.

Definition 16.1 (Linear maps as tensors)**.** Let V and W be finite-dimensional vector spaces, and let $\operatorname{Hom}(V, W)$ denote the space of vector space morphisms from V to W (i.e. linear maps). The linear isomorphism

$$\begin{aligned} W \otimes V^* &\to \operatorname{Hom}(V, W), \\ w \otimes \alpha &\mapsto (V \to W,\, v \mapsto w(\alpha \cdot_V v)) \end{aligned}$$

(extended linearly to general tensors) will play a central conceptual role in the calculations employed in this paper, as it will facilitate constructions which would otherwise be awkward or difficult to express. Linear maps and appropriately typed tensor products will be identified via this isomorphism.

Given bases $v_1, \ldots, v_m \in V$ and $w_1, \ldots, w_n \in W$, and dual bases $v^1, \ldots, v^m \in V^*$ and $w^1, \ldots, w^n \in W^*$, a linear map $A\colon V \to W$ can be written under the identification in (16.1) as

$$A = A^i_j \, w_i \otimes v^j,$$

where $A^i_j = w^i \cdot_W A \cdot_V v_j \in \mathbb{R}$, and in fact $\left[A^i_j\right] \in M_{n \times m}(\mathbb{R})$ is the matrix representation of A with respect to the bases $v_1, \ldots, v_m \in V$ and $w_1, \ldots, w_n \in W$, noting that the i and j indices denote the "output" and "input" components of A respectively. Tensors are therefore the strongly typed analog of matrices, where the $W \otimes V^*$ type information is carried by the $w_i \otimes v^j$ component.

One clarifying example of the tensor formulation is the adjoint operation of the natural pairing, also known as forming the dual of a linear map. It is straightforward to show that

$$\begin{aligned} *\colon W \otimes V^* &\to V^* \otimes W, \\ w \otimes \alpha &\mapsto \alpha \otimes w, \end{aligned}$$

(where the map is extended linearly to general tensors). This is literally the tensor abstraction of the matrix transpose operation; if $A = A^i_j \, w_i \otimes \alpha^j$, then the dual A

is $A^* = A_i^j \alpha^i \otimes w_j$. The matrix of A^* is precisely the transpose of the matrix of A with respect to the relevant bases. The map $*$ itself can be written as a 4-tensor $* \in V^* \otimes W \otimes W^* \otimes V$, where $A^* = * \cdot_{W \otimes V^*} A$.

There is a notion of the natural pairing of tensor products, which implements composition and evaluation of linear maps, and can be thought of as a natural generalization of scalar multiplication in a field. If U, V, and W are each finite-dimensional vector spaces, then the bilinear form

$$\begin{aligned} (U \otimes V^*) \times (V \otimes W) &\to U \otimes \mathbb{R} \otimes W \cong U \otimes W, \\ (u \otimes \alpha, v \otimes w) &\mapsto u \otimes (\alpha \cdot_V v) \otimes w = (\alpha \cdot_V v)\, u \otimes w \end{aligned}$$

will be denoted also by the infix notation $\cdot_V$ (i.e. $(u \otimes \alpha) \cdot_V (v \otimes w) = (\alpha \cdot_V v)\, u \otimes w$). If V itself is a tensor product of n factors which are clear from context, then $\cdot_V$ may be denoted by $\cdot^n$ (think an n-fold tensor contraction). If $n = 2$, then typically $:$ is used in place of $\cdot^2$. For example, from above, $A^* = * \cdot_{W \otimes V^*} A = * : A$.

Given a permutation $\sigma \in S_n$, define a right-action by $\sigma \colon V_1 \otimes \cdots \otimes V_n \to V_{\sigma^{-1}(1)} \otimes \cdots \otimes V_{\sigma^{-1}(n)}$, mapping elements in the obvious way. For example, $(2\,3\,4)$ acting on $v_1 \otimes v_2 \otimes v_3 \otimes v_4$ puts the second factor in the third position, the third factor in the fourth position, and the fourth factor in the second, giving $v_1 \otimes v_4 \otimes v_2 \otimes v_3$. This permutation is itself a linear map and of course can be written as a tensor. However, because it is defined in terms of a right action, the "domain factors" will come on the left. Thus σ is written as a tensor of the form $V_1^* \otimes \cdots \otimes V_n^* \otimes V_{\sigma^{-1}(1)} \otimes \cdots \otimes V_{\sigma^{-1}(n)}$ (i.e. as a $2n$-tensor). Certain tensor constructions are conducive to using such permutations. In the above example, $*$ can be written as $(1\,2) \in W^* \otimes V \otimes V^* \otimes W$.

The permutation right-action also works naturally when notated using superscripts. For example, if $B \in U \otimes V \otimes W$, then

$$B^{(1\,2)} := B \cdot_{U^* \otimes V^* \otimes W^*} (1\,2) \in V \otimes U \otimes W$$

and so

$$\left(B^{(1\,2)}\right)^{(2\,3)} = \left(B \cdot_{U^*\otimes V^*\otimes W^*} (1\,2)\right) \cdot_{V^*\otimes U^*\otimes W^*} (2\,3)$$
$$= B \cdot_{U^*\otimes V^*\otimes W^*} \left((1\,2) \cdot_{V^*\otimes U^*\otimes W^*} (2\,3)\right)$$
$$= B \cdot_{U^*\otimes V^*\otimes W^*} (1\,2)\,(2\,3)$$
$$= B \cdot_{U^*\otimes V^*\otimes W^*} (1\,3\,2) \in V \otimes W \otimes U.$$

When multiplying the permutations $(1\,2)$ and $(2\,3)$ in the third line, it is important to note that they are read left-to-right, since they are acting on B on the right.

The inline cycle notation is somewhat ambiguous in isolation because the number of factors in the domain/codomain is not specified, let alone their types. This information can sometimes be inferred from context, such as from the natural pairing subscripts, as in the following examples.

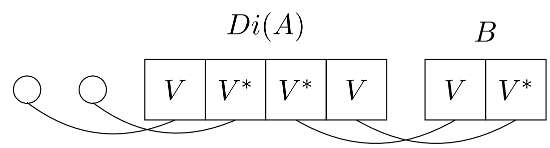
Example 16.2 (Linearizing the inversion map)**.** Let $i\colon GL(V) \to GL(V)$, $A \mapsto A^{-1}$, i.e. the linear map inversion operator, where $GL(V)$ is an open submanifold of $V \otimes V^*$ via the isomorphism $V \otimes V^* \cong \operatorname{Hom}(V, V)$. Its linearization (derivative) $Di\colon GL(V) \to V \otimes V^* \otimes (V \otimes V^*)^* \cong V \otimes V^* \otimes V^* \otimes V$ at $A \in GL(V)$ in the direction $B \in T_A(GL(V)) \cong V \otimes V^*$ is

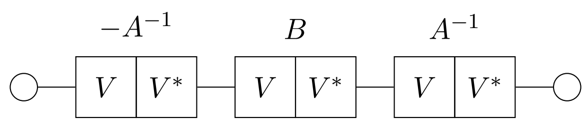
$$Di(A) \cdot_{V\otimes V^*} B = Di \cdot_{V\otimes V^*} \delta(A + \epsilon B)$$
$$= \delta\left(i(A + \epsilon B)\right)$$
$$= \delta\left((A + \epsilon B)^{-1}\right)$$
$$= \delta\left(\left(\left(1 + \epsilon B A^{-1}\right) A\right)^{-1}\right)$$
$$= \delta\left(A^{-1}\left(1 + \epsilon B A^{-1}\right)^{-1}\right)$$
$$= \delta\left(A^{-1} \sum_{n=0}^{\infty} \left(-\epsilon B A^{-1}\right)^n\right)$$

($\left|-\epsilon B A^{-1}\right|$ is taken arbitrarily small due to the derivative $\delta := \frac{d}{d\epsilon}\,|_{\epsilon=0}$ being evaluated in an arbitrarily small neighborhood of $\epsilon = 0$)

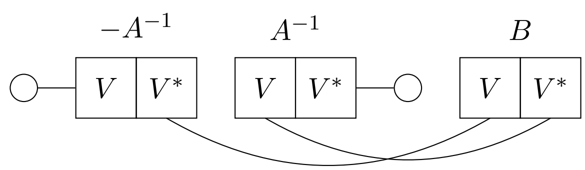
$$= \delta\left(A^{-1} - \epsilon A^{-1} B A^{-1} + O\left(\epsilon^2\right)\right)$$
$$= -A^{-1} \cdot_V B \cdot_V A^{-1}.$$

In order to "move" the B parameter out so that it plays the same syntactical role as in the original expression $Di\,(A) \cdot B$, via adjacent natural pairing, some simple tensor manipulations can be done. The process is easily and accurately expressed via diagram. The following sequence of diagrams is a sequence of equalities. The diagram should be self-explanatory, but for reference, the number of boxes for a particular label denotes the rank of the tensor, with each box labeled with its type. The lines connecting various boxes are natural pairings, and the circles represent the unpaired "slots", which comprise the type of the resulting expression.

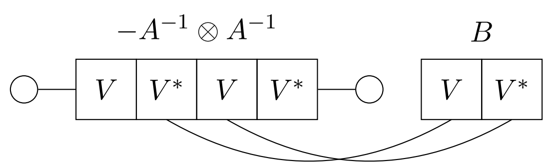




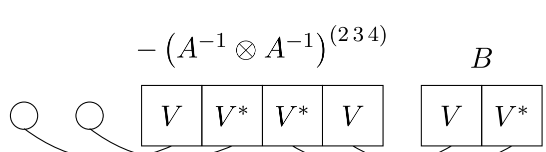
The following step is nothing but moving the boxes for B out; the natural pairings still apply to the same slots, hence the cables dangling below.



In this setting, a tensor product amounts to flippantly gluing boxes together.



In order for B to be naturally paired in the same adjacent manner as in the original expression $Di\,(A) \cdot B$, the slots of $-A^{-1} \otimes A^{-1}$ must be permuted; the second moves to the third, the third to the fourth, and the fourth to the second.



The first diagram equals the last one, thus

$$Di\,(A) \cdot_{V\otimes V^*} B = -\left(A^{-1} \otimes A^{-1}\right)^{(2\,3\,4)} \cdot_{V\otimes V^*} B,$$

and by the nondegeneracy of the natural pairing on $V \otimes V^*$, this implies that $Di\,(A) = -\left(A^{-1} \otimes A^{-1}\right)^{(2\,3\,4)}$, noting that the statement of this expression does not require the direction vector B. The permutation exponent $(2\,3\,4)$ can be calculated easily using simple tensors, if not by the above diagrammatic manipulations;

$$\begin{aligned}(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) \cdot (a_3 \otimes a_4) &= (a_1 \otimes a_4 \otimes a_2 \otimes a_3) : (b_1 \otimes b_2) \\ &= (a_1 \otimes a_2 \otimes a_3 \otimes a_4)^{(2\,3\,4)} : (b_1 \otimes b_2)\,.\end{aligned}$$

Here, the expression $(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) \cdot (a_3 \otimes a_4)$ represents the expression $A^{-1} \cdot B \cdot A^{-1}$.

The next example will later be extended to the setting of Riemannian manifolds and their metric tensor fields, and put to use to formulate what are known as harmonic maps (see (25.7)). But first, a new tensor operation must be defined.

Definition 16.3 (Parallel tensor product)**.** If U, V, W, X are vector spaces and $A \in U \otimes V$ and $B \in W \otimes X$, then define their **parallel tensor product** $A \boxtimes B$ by

$$A \boxtimes B := (A \otimes B)^{(2\,3)} \in (U \otimes W) \otimes (V \otimes X)\,.$$

The parentheses in the type specification are unnecessary, but hint at what the tensor decomposition for the quantity $A \boxtimes B$ should be, if used as an operand to $\boxtimes$ again (see below).

If A and B represent linear maps, then $A \boxtimes B \in (U \otimes W) \otimes (V \otimes X)$ represents their tensor product as linear maps (the parentheses are unnecessary but hint at what the domain and codomain are, and for use of $A \boxtimes B$ as an operand in another parallel tensor product), which is a "parallel" composition; if $\alpha \in V^*$ and $\beta \in X^*$, then $(A \boxtimes B) \cdot_{V^* \otimes X^*} (\alpha \otimes \beta) = (A \cdot_{V^*} \alpha) \otimes (B \cdot_{X^*} \beta)$.

There is a slight ambiguity in the notation coming from a lack of specification on how the tensor product of the operands is decomposed in the case when there is more than one such decomposition. Notation explicitly resolving this ambiguity will not be needed in this paper as the relevant tensor product is usually clear from context.

The parallel tensor product is associative; if Y and Z are also vector spaces and $C \in Y \otimes Z$, then

$$(A \boxtimes B) \boxtimes C = A \boxtimes (B \boxtimes C) \in (U \otimes W \otimes Y) \otimes (V \otimes X \otimes Z)\,,$$

allowing multiply-parallel tensor products.

Example 16.4 (Tensor product of inner product spaces)**.** If (V, g) and (W, h) are inner product spaces (noting that $g \in V^* \otimes V^*$ and $h \in W^* \otimes W^*$ are symmetric, i.e. literally invariant under $(1\,2)$), then $W \otimes V^*$ is an inner product space having induced inner product $k\,(A, B) := \mathrm{Tr}_V \left(g^{-1} \cdot_{V^*} A^* \cdot_{W^*} h \cdot_W B\right)$. Here, the "inputs" of A and B (the V^* factors) are being paired using $g^{-1} \in V \otimes V$, while the "outputs" (the W factors) are being paired using $h \in W^* \otimes W^*$, and the trace is used to "complete the cycle" by plugging the output into the input, thereby producing a real number. The expression $k\,(A, B)$ can be written in a more natural way, which takes advantage of the linear composition, as $A : k : B$ (or, pedantically, $A \cdot_{W^* \otimes V} k \cdot_{W \otimes V^*} B$), instead of the more common but awkward trace expression mentioned earlier. In the tensor formalism, the inner product k should have type $W^* \otimes V \otimes W^* \otimes V$. Permuting the middle two components of the 4-tensor $h \otimes g^{-1} \in W^* \otimes W^* \otimes V \otimes V$ gives the correct type. In fact, $k = h \boxtimes g^{-1}$. A further advantage to this formulation is that if any or all of A, k, B are functions, there is a clear product rule for derivatives of the expression $A : k : B$. This is something that is used critically in Riemannian geometry in the form of covariant derivatives of tensor fields (see (21.2)).

In this paper, the main use of the tensor formulation of linear maps is twofold: to facilitate linear algebraic constructions which would otherwise be difficult or awkward (this includes the ability to express derivatives of [possibly vector or manifold-valued] maps without needing to "plug in" the derivative's directional argument), and to make clear the product-rule behavior of many important differentiable constructions.

17 Bundle Constructions

In order to use the calculus of variations involving Lagrangians depending tangent maps of maps between smooth manifolds, it suffices to consider Lagrangians defined on smooth vector bundle morphisms. Continuing in the style of the previous section, a "full" tensor product of smooth vector bundles (17.4) will be formulated which will then allow expression of smooth vector bundle morphisms as tensor fields, sometimes called two-point tensor fields (see [MH83, pg. 70]). The full arsenal of tensor calculus can then be used to considerable advantage.

First, some definitions and simpler bundle constructions will be introduced. A **smooth [fiber] bundle** (hereafter referred to simply as a smooth bundle) is a 4-tuple $(\mathcal{E}, E, \pi, N)$ where $\mathcal{E}$, E and N are smooth manifolds and $\pi\colon E \to N$ is locally trivial, i.e. N is covered by open sets $\{U_\alpha\}$ such that $\pi^{-1}(U_\alpha) \cong U_\alpha \times \mathcal{E}$ as smooth manifolds. The manifolds $\mathcal{E}$, E and N are called the **typical fiber**, the **total space**, and the **base space** respectively. The map π is called the **bundle projection**. The full 4-tuple specifying a bundle can be recovered from the bundle projection map, so a locally trivial smooth map can be said to define a smooth bundle. The dimension of the typical fiber of a bundle will be called its **rank**, and will be denoted by $\operatorname{Rank}\pi$ or $\operatorname{Rank} E$ when the bundle is understood from context.

The **space of smooth sections** of a smooth bundle defined by $\pi\colon E \to N$ is

$$\Gamma(\pi) := \{\sigma \in C^\infty(N, E) \mid \pi \circ \sigma = \mathrm{Id}_N\},$$

and may also be denoted by $\Gamma(E)$, if the bundle is clear from context. If nonempty, $\Gamma(\pi)$ is generally an infinite-dimensional manifold (the exception being when the base space N is finite).

Proposition 17.1 (Trivial bundle)**.** *Let M and N be smooth manifolds. With $M \mathbin{\times\!\!\!\!\rightarrow} N := M \times N$ and*

$$\pi^{M \mathbin{\times\!\!\!\!\rightarrow} N} := \mathrm{Pr}_2^{M\times N}\colon M \mathbin{\times\!\!\!\!\rightarrow} N \to N$$

*defines a smooth bundle $\left(M, M \mathbin{\times\!\!\!\!\rightarrow} N, \pi^{M \mathbin{\times\!\!\!\!\rightarrow} N}, N\right)$, called a **trivial bundle**. Similarly, with $M \mathbin{\leftarrow\!\!\!\!\times} N := M \times N$ and $\pi^{M \mathbin{\leftarrow\!\!\!\!\times} N} := \mathrm{Pr}_1^{M\times N}\colon M \mathbin{\leftarrow\!\!\!\!\times} N \to M$, $\left(N, M \mathbin{\leftarrow\!\!\!\!\times} N, \pi^{M \mathbin{\leftarrow\!\!\!\!\times} N}, M\right)$ is a trivial bundle.*

No proof is deemed necessary for (17.1), as each bundle projection trivializes globally in the obvious way. The $\mathbin{\times\!\!\!\!\rightarrow}$ symbol is a composite of $\times$ (indicating direct product) and $\to$ or $\leftarrow$ (indicating the base space).

If M and N are smooth manifolds as in (17.1), then there are two particularly useful natural identifications.

$$
\begin{aligned}
C^{\infty}(M, N) &\cong \Gamma(M \mathbin{\times\!\!\!\!\leftarrow} N) & \qquad C^{\infty}(M, N) &\cong \Gamma(N \mathbin{\times\!\!\!\!\rightarrow} M) \\
\phi &\mapsto \mathrm{Id}_M \times_M \phi & \phi &\mapsto \phi \times_M \mathrm{Id}_M \\
\mathrm{Pr}_2^{M \times N} \circ \Phi &\mapsfrom \Phi & \mathrm{Pr}_1^{N \times M} \circ \Phi &\mapsfrom \Phi
\end{aligned}
$$

These identifications can be thought of identifying a map $\phi \in C^{\infty}(M, N)$ with its graph in $M \times N$ and $N \times M$ respectively. Furthermore, this allows bundle theory to be applied to reasoning about spaces of maps. The symbols $M \mathbin{\times\!\!\!\!\leftarrow} N$ and $N \mathbin{\times\!\!\!\!\rightarrow} M$ now carry a significant amount of meaning. Generally $N \mathbin{\times\!\!\!\!\rightarrow} M$ will be used in this paper, for consistency with the $\mathrm{Hom}(V, W) \cong W \otimes V^*$ convention discussed in Section 16. The symbols $\mathbin{\times\!\!\!\!\rightarrow}$ and $\mathbin{\times\!\!\!\!\leftarrow}$ are examples of telescoping notation, as they are built notationally on $\times$, and conceptually on the direct product, which is what is denoted by $\times$. The arrow portion of the symbols can be discarded when type-specificity is not needed.

Proposition 17.2 (Direct product bundle)**.** *Let $(\mathcal{E}, E, \pi^E, M)$ and $(\mathcal{F}, F, \pi^F, N)$ be smooth bundles. Then*

$$\pi^E \times \pi^F \colon E \times F \to M \times N,\ (e, f) \mapsto \left(\pi^E(e), \pi^F(f)\right)$$

defines a smooth bundle $(\mathcal{E} \times \mathcal{F}, E \times F, \pi^E \times \pi^F, M \times N)$. This bundle is called the ***direct product of*** π^E ***and*** π^F*, and is not necessarily a trivial bundle.*

Proof. Let $\Psi^E \colon \left(\pi^E\right)^{-1}(U) \to U \times \mathcal{E}$ and $\Psi^F \colon \left(\pi^F\right)^{-1}(V) \to V \times \mathcal{F}$ trivialize π^E and π^F over open sets $U \subseteq M$ and $V \subseteq N$ respectively. Then

$$\Psi^E \times \Psi^F \colon \left(\pi^E\right)^{-1}(U) \times \left(\pi^F\right)^{-1}(V) \to U \times \mathcal{E} \times V \times \mathcal{F}$$

has inverse $\left(\Psi^E\right)^{-1} \times \left(\Psi^F\right)^{-1}$. Note that

$$
\begin{aligned}
\left(\pi^E\right)^{-1}(U) \times \left(\pi^F\right)^{-1}(V) &= \left\{(e, f) \in E \times F \mid \pi^E(e) \in U,\ \pi^F(f) \in V\right\} \\
&= \left\{(e, f) \in E \times F \mid \left(\pi^E \times \pi^F\right)(e, f) \in U \times V\right\} \\
&= \left(\pi^E \times \pi^F\right)^{-1}(U \times V),
\end{aligned}
$$

and that

$$P \colon (U \times \mathcal{E}) \times (V \times \mathcal{F}) \to (U \times V) \times (\mathcal{E} \times \mathcal{F}),\ ((u, e), (v, f)) \mapsto ((u, v), (e, f))$$

defines a diffeomorphism. Then

$$\Psi^{E\times F} := P \circ \left(\Psi^E \times \Psi^F\right) \colon \left(\pi^E \times \pi^F\right)^{-1}(U \times V) \to (U \times V) \times (\mathcal{E} \times \mathcal{F})$$

defines a diffeomorphism, and

$$\begin{aligned}
&\mathrm{Pr}_1^{(U\times V)\times(\mathcal{E}\times\mathcal{F})} \circ \Psi^{E\times F}(e,f) \\
&= \mathrm{Pr}_1^{(U\times V)\times(\mathcal{E}\times\mathcal{F})} \circ P \circ \left(\Psi^E \times \Psi^F\right)(e,f) \\
&= \mathrm{Pr}_1^{(U\times V)\times(\mathcal{E}\times\mathcal{F})} \circ P\left(\Psi^E(e), \Psi^F(f)\right) \\
&= \mathrm{Pr}_1^{(U\times V)\times(\mathcal{E}\times\mathcal{F})} \circ P\left(\Psi^E(e), \Psi^F(f)\right) \\
&= \mathrm{Pr}_1^{(U\times V)\times(\mathcal{E}\times\mathcal{F})} \\
&\quad \circ P\left(\left(\mathrm{Pr}_1^{U\times\mathcal{E}} \circ \Psi^E(e), \mathrm{Pr}_2^{U\times\mathcal{E}} \circ \Psi^E(e)\right), \left(\mathrm{Pr}_1^{V\times\mathcal{F}} \circ \Psi^F(f), \mathrm{Pr}_2^{V\times\mathcal{F}} \circ \Psi^F(f)\right)\right) \\
&= \mathrm{Pr}_1^{(U\times V)\times(\mathcal{E}\times\mathcal{F})} \\
&\quad \left(\left(\mathrm{Pr}_1^{U\times\mathcal{E}} \circ \Psi^E(e), \mathrm{Pr}_1^{V\times\mathcal{F}} \circ \Psi^F(f)\right), \left(\mathrm{Pr}_2^{U\times\mathcal{E}} \circ \Psi^E(e), \mathrm{Pr}_2^{V\times\mathcal{F}} \circ \Psi^F(f)\right)\right) \\
&= \left(\mathrm{Pr}_1^{U\times\mathcal{E}} \circ \Psi^E(e), \mathrm{Pr}_1^{V\times\mathcal{F}} \circ \Psi^F(f)\right) \\
&= \left(\pi^E(e), \pi^F(f)\right) \\
&= \left(\pi^E \times \pi^F\right)(e,f),
\end{aligned}$$

showing that $\Psi^{E\times F}$ trivializes $\pi^E \times \pi^F$ over $U \times V \subseteq M \times N$. Since $M \times N$ can be covered by such trivializing sets, this establishes that $\pi^E \times \pi^F$ defines a smooth bundle. The typical fiber of $\pi^E \times \pi^F$ is $\mathcal{E} \times \mathcal{F}$. □

A **smooth vector bundle** is a fiber bundle whose typical fiber is a vector space and whose local trivializations are linear isomorphisms when restricted to each fiber. If $(\mathcal{E}, E, \pi, M)$ is a smooth vector bundle, then its **dual vector bundle** $(\mathcal{E}^*, E^*, \pi^*, M)$ is a smooth vector bundle defined in the following way.

$$E^* := \coprod_{p\in M} (E_p)^*, \qquad \pi^* \colon E^* \to M,\ \eta_p \mapsto p.$$

Because $\mathcal{E}$ is a vector space, the notation $\mathcal{E}^*$ is already defined. In analogy with Section 16, there are natural pairings on a vector bundle and its dual, defined simply by evaluation. If $p \in M$, $\eta \in E_p^*$ and $e \in E_p$, then $\eta \cdot_E e := \eta \cdot_{E_p} e$ and $e \cdot_E \eta := e \cdot_{E_p} \eta$. Both expressions evaluate to $\eta(e)$. Natural traces and n-fold tensor contraction can be defined analogously. Again, while seemingly pedantic, the subscripted natural pairing notation will prove to be a valuable tool in articulating and error-checking calculations involving vector bundles. To generalize the rest of Section 16 will require the definition of additional structures.

For the remainder of this section, let $\left(\mathcal{E}, E, \pi^E, M\right)$ and $\left(\mathcal{F}, F, \pi^F, N\right)$ now be smooth *vector* bundles. The following construction is essentially an alternate notation for $\pi^E \times \pi^F \colon E \times F \to M \times N$, but is one that takes advantage of the fact that π^E and π^F are vector bundles, and encodes in the notation the fact that the resulting construction is also a vector bundle. This is analogous to how $V \times W$ is a vector space with a natural structure if V and W are vector spaces, except that this is usually denoted by $V \oplus W$.

Proposition 17.3 ("Full" direct sum vector bundle)**.** *If*

$$E \oplus_{M \times N} F := E \times F,$$

Then

$$\pi^E \oplus_{M \times N} \pi^F := \pi^E \times \pi^F \colon E \oplus_{M \times N} F \to M \times N$$

defines a smooth vector bundle $\left(\mathcal{E} \oplus \mathcal{F}, E \oplus_{M \times N} F, \pi^E \oplus_{M \times N} \pi^F, M \times N\right)$*, called the* ***full direct sum of*** π^E ***and*** π^F*.*

For each $(p, q) \in M \times N$*, the vector space structure on* $\left(\pi^E \oplus_{M \times N} \pi^F\right)^{-1}(p, q)$ *is given in the following way. Let* $\alpha \in \mathbb{R}$ *and* $(e_1, f_1), (e_2, f_2) \in \left(\pi^E \oplus_{M \times N} \pi^F\right)^{-1}(p, q)$*. Then*

$$\alpha (e_1, f_1) + (e_2, f_2) = (\alpha e_1 + e_2, \alpha f_1 + f_2).$$

It is critical to see (17.5) for remarks on notation.

Proof. Let U, V, $\mathcal{E}$, $\mathcal{F}$, P, Ψ^E, Ψ^F and $\Psi^{E \times F}$ be as in the proof of (17.2), and define $\Psi^{E \oplus_{M \times N} F} := \Psi^{E \times F}$. Noting that $\Psi^{E \oplus_{M \times N} F}$ is a smooth bundle isomorphism over $\mathrm{Id}_{U \times V}$, so to show that $\Psi^{E \oplus_{M \times N} F}$ is a linear isomorphism in each fiber, it suffices to show that it is linear in each fiber. Let $\alpha \in \mathbb{R}$, $(p, q) \in U \times V$ and $(e_1, f_1), (e_2, f_2) \in$

$\left(\pi^E \oplus_{M\times N} \pi^F\right)^{-1}(p,q)$. Then

$$
\begin{aligned}
&\Psi^{E\oplus_{M\times N}F}\left(\alpha e_1 + e_2, \alpha f_1 + f_2\right)\\
&= P\circ\left(\Psi^E \times \Psi^F\right)\left(\alpha e_1 + e_2, \alpha f_1 + f_2\right)\\
&= P\left(\Psi^E\left(\alpha e_1 + e_2\right), \Psi^F\left(\alpha f_1 + f_2\right)\right)\\
&= P\left(\alpha\,\Psi^E\left(e_1\right) + \Psi^E\left(e_2\right), \alpha\,\Psi^F\left(f_1\right) + \Psi^F\left(f_2\right)\right)\\
&\quad \text{(by trivial vector bundle structures on } U\times\mathcal{E} \text{ and } V\times\mathcal{F}\text{)}\\
&= \alpha\,P\left(\Psi^E\left(e_1\right), \Psi^F\left(f_1\right)\right) + P\left(\Psi^E\left(e_2\right), \Psi^F\left(f_2\right)\right)\\
&\quad \text{(by trivial vector bundle structure on } (U\times V)\times(\mathcal{E}\times\mathcal{F})\text{)}\\
&= \alpha\,P\circ\left(\Psi^E\times\Psi^F\right)\left(e_1, f_1\right) + P\circ\left(\Psi^E\times\Psi^F\right)\left(e_2, f_2\right)\\
&= \alpha\,\Psi^{E\oplus_{M\times N}F}\left(e_1, f_1\right) + \Psi^{E\oplus_{M\times N}F}\left(e_2, f_2\right).
\end{aligned}
$$

Thus $\Psi^{E\oplus_{M\times N}F}$ is linear in each fiber, and because it is invertible, it is a linear isomorphism in each fiber. In particular, $\Psi^{E\oplus_{M\times N}F}$ is a smooth vector bundle isomorphism over $\mathrm{Id}_{U\times V}$. Applying $\left(\Psi^{E\oplus_{M\times N}F}\right)^{-1}$ to the above equation gives

$$\left(\alpha e_1 + e_2, \alpha f_1 + f_2\right) = \alpha\left(e_1, f_1\right) + \left(e_2, f_2\right),$$

as desired. □

This construction differs from the Whitney sum of two vector bundles, as the base spaces of the bundles are kept separate, and aren't even required to be the same. This allows the identification of $T(M\times N)\to M\times N$ as $TM\oplus_{M\times N} TN\to M\times N$, which may be done without comment later in this paper. Some important related structures are $\mathrm{Pr}_1^*\pi_M^{TM}\colon \mathrm{Pr}_1^* TM\to M\times N$ and $\mathrm{Pr}_2^*\pi_N^{TN}\colon \mathrm{Pr}_2^* TN\to M\times N$, where $\mathrm{Pr}_i := \mathrm{Pr}_i^{M\times N}$.

The next construction is what will be used in the implementation of smooth vector bundle morphisms as tensor fields.

Proposition 17.4 ("Full" tensor product bundle). *If*

$$E\otimes_{M\times N} F := \coprod_{(p,q)\in M\times N} E_p\otimes F_q \ \textit{(disjoint union)},$$

Then

$$\pi^E\otimes_{M\times N}\pi^F\colon E\otimes_{M\times N} F\to M\times N,\ \alpha^{ij} e_i\otimes f_j\mapsto\left(\pi^E\left(e_1\right), \pi^F\left(f_1\right)\right)\ \textit{(here, } \alpha^{ij}\in\mathbb{R}\textit{)}$$

defines a smooth vector bundle $\left(\mathcal{E}\otimes\mathcal{F}, E\otimes_{M\times N}F, \pi^E\otimes_{M\times N}\pi^F, M\times N\right)$*, called the* ***full tensor product***[19] ***of*** π^E ***and*** π^F.

It is critical to see (17.5) for remarks on notation.

Proof. Since the argument $\alpha^{ij}e_i\otimes f_j$ in the definition of $\pi^E\otimes_{M\times N}\pi^F$ is not necessarily unique, the well-definedness of $\pi^E\otimes_{M\times N}\pi^F$ must be shown. Let $\alpha^{ij}e_i^1\otimes f_j^1=\beta^{ij}e_i^2\otimes f_j^2$. Then in particular, $\alpha^{ij}e_i^1\otimes f_j^1, \beta^{ij}e_i^2\otimes f_j^2\in E_p\otimes F_q$ for some $(p,q)\in M\times N$, and therefore $e_i^1, e_i^2\in E_p$ and $f_j^1, f_j^2\in F_q$ for each index i and j. Thus $\pi^E\left(e_1^1\right)=p=\pi^E\left(e_1^2\right)$ and $\pi^F\left(f_1^1\right)=q=\pi^F\left(f_1^2\right)$, so the expression defining $\pi^E\otimes_{M\times N}\pi^F$ is well-defined.

The set $E\otimes_{M\times N}F$ does not have an a priori global smooth manifold structure, as it is defined as the disjoint union of vector spaces. A smooth manifold structure compatible with that of the constituent vector spaces will now be defined.

Let $\Psi^E\colon\left(\pi^E\right)^{-1}(U)\to U\times\mathcal{E}$ and $\Psi^F\colon\left(\pi^F\right)^{-1}(V)\to V\times\mathcal{F}$ trivialize π^E and π^F over open sets $U\subseteq M$ and $V\subseteq N$ respectively, such that Ψ^E and Ψ^F are each linear in each fiber. Define

$$\Psi^{E\otimes_{M\times N}F}\colon\left(\pi^E\otimes_{M\times N}\pi^F\right)^{-1}(U\times V)\quad\to\quad(U\times V)\times(\mathcal{E}\otimes\mathcal{F})$$

by

$$\Psi^{E\otimes_{M\times N}F}(X):=\left(\left(\pi^E\otimes_{M\times N}\pi^F\right)(X),\left(\left(\mathrm{Pr}_2^{U\times\mathcal{E}}\circ\Psi^E\right)\otimes\left(\mathrm{Pr}_2^{V\times\mathcal{F}}\circ\Psi^F\right)\right)(X)\right).$$

The map $\Psi^{E\otimes_{M\times N}F}$ is well-defined and smooth in each fiber by construction, since for each $(p,q)\in U\times V$,

$$\left(\mathrm{Pr}_2^{U\times\mathcal{E}}\circ\Psi^E\right)\otimes\left(\mathrm{Pr}_2^{V\times\mathcal{F}}\circ\Psi^F\right)|_{E_p\otimes E_q}\colon E_p\otimes E_q\to\mathcal{E}\otimes\mathcal{F}$$

is a linear isomorphism by construction. Additionally, $\Psi^{E\otimes_{M\times N}F}$ has been constructed so that

$$\mathrm{Pr}_1^{(U\times V)\times(\mathcal{E}\otimes_{M\times N}\mathcal{F})}\circ\Psi^{E\otimes_{M\times N}F}=\pi^E\otimes_{M\times N}\pi^F$$

on $\left(\pi^E\otimes_{M\times N}\pi^F\right)^{-1}(U\times V)$. Define the smooth structure on

$$\left(\pi^E\otimes_{M\times N}\pi^F\right)^{-1}(U\times V)\subseteq E\otimes_{M\times N}F$$

by declaring $\Psi^{E\otimes_{M\times N}F}$ to be a diffeomorphism. The map $\pi^E\otimes_{M\times N}\pi^F$ is trivialized over $U\times V$. The set $E\otimes_{M\times N}F$ can be covered by such trivializing open sets. Thus

[19] This construction is alluded to in [KMS93, pg. 121], but is not defined or discussed.

$E \otimes_{M\times N} F$ has been shown to be locally diffeomorphic to the direct product of smooth manifolds, and therefore it has been shown to be a smooth manifold. With respect to the smooth structure on $E \otimes_{M\times N} F$, the map $\pi^E \otimes_{M\times N} \pi^F$ is smooth, and has therefore been shown to define a smooth vector bundle. $\square$

Remark 17.5 (Notation regarding base space). The "full" direct sum (17.3) and "full" tensor product (17.4) bundle constructions allow direct sums and tensor products to be taken of vector bundles when the base spaces differ. If the base spaces are the same, then the construction "joins" them, producing a vector bundle over that shared base space. For example, if E and F are vector bundles over M, then $E \otimes_{M\times M} F$ has base space $M \times M$, while $E \otimes F$ has base space M. The base space can be specified in either case as a notational aide; the latter example would be written as $E \otimes_M F$. If no subscript is provided on the $\otimes$ symbol, then the base spaces are "joined" if possible (if they are the same space), otherwise they are kept separate, as in the "full" tensor product construction. This notational convention conforms to the standard Whitney sum and tensor product bundle notation, and uses the notion of telescoping notation to provide more specificity when necessary.

Given a fiber bundle, a natural vector bundle can be constructed "on top" of it, essentially quantifying the variations of bundle elements along each fiber. This is known as the vertical bundle, and it plays a critical role in the development of Ehresmann connections, which provide the "horizontal complement" to the vertical bundle.

Proposition 17.6 (Vertical bundle)**.** *Let $\pi^E \colon E \to M$ define a smooth [fiber] bundle. If $VE := \ker T\pi^E \leq TE$, then $\pi^{VE} := \pi_E^{TE}|_{VE} \colon VE \to E$ defines a smooth vector bundle subbundle of $\pi_E^{TE} \colon TE \to E$, called the **vertical bundle** over E. Furthermore, the fiber over $e \in E$ is $V_e E = T_e E_{\pi^E(e)} \leq T_e E$.*

Proof. Because π^E is a smooth surjective submersion, $VE \to E$ is a subbundle of $TE \to E$ having corank $\dim M$ and therefore rank equal to that of E. Furthermore, if $e \in E$ and $\epsilon \mapsto e_\epsilon \in E_{\pi^E(e)}$, then δe_ϵ represents an arbitrary element of $T_e E_{\pi^E(e)}$, and $T\pi^E(\delta e_\epsilon) = \delta\left(\pi^E(e_\epsilon)\right) = \delta(\pi(e)) = 0$, showing that $\delta e_\epsilon \in \ker T\pi^E$, and therefore that $\delta e_\epsilon \in V_e E$. This shows that $T_e E_{\pi^E(e)} \subseteq V_e E$. Because $\dim T_e E_{\pi^E(e)} = \operatorname{Rank} E$, this shows that $T_e E_{\pi^E(e)} = V_e E$. $\square$

Given the extra structure that a vector bundle provides over a [fiber] bundle, there is a canonical smooth vector bundle isomorphism which adds significant value to the pullback bundle formalism used throughout this paper. This can be seen put to greatest use in Part IV, for example, in development of the first variation (see (25.1)).

Proposition 17.7 (Vertical bundle as pullback)**.** *If* $\pi\colon E \to M$ *defines a smooth vector bundle, then*

$$\begin{aligned} \iota_{VE}^{\pi^* E}\colon \pi^* E &\to VE, \\ (x, y) &\mapsto \delta_\epsilon (x + \epsilon y) \end{aligned}$$

is a smooth vector bundle isomorphism over Id_E*, called the* ***vertical lift****, having inverse*

$$\iota_{\pi^* E}^{VE}\colon \delta e_\epsilon \mapsto \left(e_0, \lim_{\epsilon \to 0} \frac{e_\epsilon - e_0}{\epsilon} \right),$$

where, without loss of generality, e_ϵ *is an* E*-valued variation which lies entirely in a single fiber.*

Proof. It is clear that $\iota_{VE}^{\pi^* E}$ is linear and injective on each fiber. By a dimension counting argument, it is therefore an isomorphism on each fiber. Because it preserves the base-point, it is a vector bundle isomorphism over Id_E. Because the map $(x, y, \epsilon) \mapsto x + \epsilon y$ is smooth, so is the defining expression for $\iota_{VE}^{\pi^* E}$, thereby establishing smoothness. That $\iota_{\pi^* E}^{VE}$ inverts $\iota_{VE}^{\pi^* E}$ is a trivial calculation. □

18 Strongly-Typed Tensor Field Operations

Because vector bundles and the related operations can be thought of conceptually as "sheaves of linear algebra", the constructions in Section 16, generalized earlier in this section, can be further generalized to the setting of sections of vector bundles.

If E, F, G are smooth vector bundles over M, then define the natural pairing of a tensor field with a vector:

$$\begin{aligned} \cdot_F\colon \Gamma (E \otimes_M F^*) \times F &\to E, \\ (e \otimes_M \phi, f) &\mapsto e\left(\pi^F (f)\right) \left[\phi\left(\pi^F (f)\right) \cdot_F f\right], \end{aligned}$$

extending linearly to general tensor fields. Further, define the natural pairing of tensor fields:

$$\begin{aligned} \cdot_F \colon \Gamma(E \otimes_M F^*) \times \Gamma(F \otimes_M G) &\to \Gamma(E \otimes_M G), \\ (e \otimes_M \phi, f \otimes_M g) &\mapsto \big(p \mapsto e(p) \otimes_M \big(\phi(p) \cdot_{F_p} f(p)\big) \otimes_M g(p)\big) \\ &= \big(p \mapsto \big(\phi(p) \cdot_{F_p} f(p)\big)(e \otimes_M g)(p)\big), \end{aligned}$$

extending linearly to general tensor fields. This multiple use of the $\cdot_F$ symbol is a concept known as **operator overloading** in computer programming. No ambiguity is caused by this overloading, as the particular use can be inferred from the types of the operands. As before, the subscript F may be optionally omitted when clear from context.

The permutations defined in Section 16 are generalized as tensor fields. If $F_1, \ldots, F_n$ are smooth vector bundles over M, and $\sigma \in S_n$ is a permutation, then σ can act on $F_1 \otimes_M \cdots \otimes_M F_n$ by permuting its factors, and therefore can be identified with a tensor field

$$\sigma \in \Gamma\left(F_1^* \otimes_M \cdots \otimes_M F_n^* \otimes_M F_{\sigma^{-1}(1)} \otimes_M \cdots \otimes_M F_{\sigma^{-1}(n)}\right)$$

defined by

$$(f_1 \otimes_M \cdots \otimes_M f_n) \cdot_{F_1^* \otimes_M \cdots \otimes_M F_n^*} \sigma := f_{\sigma^{-1}(1)} \otimes_M \cdots \otimes_M f_{\sigma^{-1}(n)}.$$

An important feature of such permutation tensor fields is that they are parallel with respect to covariant derivatives on the factors $F_1, \ldots, F_n$ (see (21.12) for more on this).

19 Pullback Bundles

The pullback bundle, defined below, is a crucial building block for many important bundle constructions, as it enriches the type system dramatically, and allows the tensor formulation of linear algebra to be extended to the vector bundle setting. In particular, the abstract, global formulation of the space of smooth vector bundle morphisms over a map $\phi \colon M \to N$ is achieved quite cleanly using a pullback bundle. Furthermore, the use of pullback bundles and pullback covariant derivatives simplifies what would otherwise be local coordinate calculations, thereby giving more insight into the geometric structure of the problem.

For the duration of this section, let $(\mathcal{F}, F, \pi, N)$ be a smooth bundle having rank r.

Proposition 19.1 (Pullback bundle)**.** *Let M and N be smooth manifolds and let $\phi\colon M \to N$ be smooth. If*

$$\phi^* F := \{(m, f) \in M \times F \mid \phi(m) = \pi(f)\},$$

and

$$\pi^{\phi^* F} := \operatorname{Pr}_1^{M \times F} |_{\phi^* F}\colon \phi^* F \to M,\ (m, f) \mapsto m,$$

*then $\left(\mathcal{F}, \phi^* F, \pi^{\phi^* F}, M\right)$ defines a smooth bundle. In particular, $\phi^* F$ is a smooth manifold having dimension $\dim M + \operatorname{Rank} \pi$. The bundle defined by $\pi^{\phi^* F}$ is called the **pullback of π by ϕ**.*

Proof. Recalling that $\mathcal{F}$ denotes the typical fiber of π, let $\Psi\colon \pi^{-1}(U) \to U \times \mathcal{F}$ trivialize π over open set $U \subseteq N$. Define

$$\Psi_\phi\colon \phi^* \left(\pi^{-1}(U)\right) \to \phi^{-1}(U) \times \mathcal{F},\ (m, f) \mapsto \left(m, \operatorname{Pr}_2^{U \times \mathcal{F}} \circ \Psi(f)\right)$$

and

$$\Psi_\phi^{-1}\colon \phi^{-1}(U) \times \mathcal{F} \to \phi^* \left(\pi^{-1}(U)\right),\ (m, f) \mapsto \left(m, \Psi^{-1}(\phi(m), f)\right).$$

Claim (1): Ψ_ϕ and Ψ_ϕ^{-1} are smooth. Proof: $\phi^* \left(\pi^{-1}(U)\right) \subseteq \phi^{-1}(U) \times \pi^{-1}(U)$, and Ψ_ϕ is clearly smooth as a map defined on the larger manifold. Therefore it restricts to a smooth map on $\phi^* \left(\pi^{-1}(U)\right)$. An analogous argument shows that Ψ_ϕ^{-1} is smooth. Claim (1) proved.

Claim (2): Ψ_ϕ^{-1} inverts Ψ_ϕ. Proof: Let $(m, f) \in \phi^* \left(\pi^{-1}(U)\right)$. Then

$$\begin{aligned}
\Psi_\phi^{-1} \circ \Psi_\phi(m, f) &= \Psi_\phi^{-1} \left(m, \operatorname{Pr}_2^{U \times \mathcal{F}} \circ \Psi(f)\right) \\
&= \left(m, \Psi^{-1} \left(\phi(m), \operatorname{Pr}_2^{U \times \mathcal{F}} \circ \Psi(f)\right)\right) \\
&= \left(m, \Psi^{-1} \left(\pi(f), \operatorname{Pr}_2^{U \times \mathcal{F}} \circ \Psi(f)\right)\right) \text{ (since } \phi(m) = \pi(f)) \\
&= \left(m, \Psi^{-1} \left(\operatorname{Pr}_1^{U \times \mathcal{F}} \circ \Psi(f), \operatorname{Pr}_2^{U \times \mathcal{F}} \circ \Psi(f)\right)\right) \\
&= \left(m, \Psi^{-1} \circ \Psi(f)\right) \\
&= (m, f).
\end{aligned}$$

With $g \in \mathcal{F}$,

$$\begin{aligned}\Psi_\phi \circ \Psi_\phi^{-1}(m, g) &= \Psi_\phi\left(m, \Psi^{-1}(\phi(m), g)\right)\\ &= \left(m, \mathrm{Pr}_2^{U\times\mathcal{F}} \circ \Psi \circ \Psi^{-1}(\phi(m), g)\right)\\ &= \left(m, \mathrm{Pr}_2^{U\times\mathcal{F}}(\phi(m), g)\right)\\ &= (m, g),\end{aligned}$$

proving Claim (2).

Claim (3): Ψ_ϕ trivializes $\pi^{\phi^* F}$ over $\phi^{-1}(U) \subseteq M$. Proof: Let $(m, f) \in \phi^*\left(\pi^{-1}(U)\right)$. Then

$$\mathrm{Pr}_1^{\phi^{-1}(U)\times\mathcal{F}} \circ \Psi_\phi(m, f) = \mathrm{Pr}_1^{\phi^{-1}(U)\times\mathcal{F}} \circ \left(m, \mathrm{Pr}_2^{U\times\mathcal{F}} \circ \Psi(f)\right) = m = \pi^{\phi^* F}(m, f),$$

and by claims (1) and (2), Ψ_ϕ is a diffeomorphism, so Ψ_ϕ trivializes $\pi^{\phi^* F}$ over $\phi^{-1}(U) \subseteq M$. Claim (3) proved.

Since M can be covered with sets as in claim (3) and since the typical fiber of $\pi^{\phi^* F}$ is diffeomorphic to $\mathcal{F}$, this shows that $\pi^{\phi^* F}$ defines a smooth bundle

$$\left(\mathcal{F}, \phi^* F, \pi^{\phi^* F}, M\right).$$

Because $\phi^* F$ is locally diffeomorphic to the product of an open subset of M with $\mathcal{F}$, $\phi^* F$ has been shown to be a smooth manifold having dimension $\dim M + \dim \mathcal{F} = \dim M + \mathrm{Rank}\,\pi$. □

While the pullback bundle is constructed as a submanifold of a direct product, there is a natural bundle morphism into the pulled-back bundle, which serves as an interface to maps defined on the pulled-back bundle. Usually this morphism is notationally suppressed, just as naturally isomorphic spaces can be identified without explicit notation.

Corollary 19.2 (Pullback fiber projection bundle morphism)**.** *If $\phi\colon M \to N$ is smooth, then*

$$\begin{aligned}\rho_F^{\phi^* F}\colon \phi^* F &\to F,\\ (m, f) &\mapsto f\end{aligned}$$

is a smooth bundle morphism over ϕ which is an isomorphism when restricted to any fiber of $\phi^* F$.

Because $\rho_F^{\phi^* F}$ is the projection $\mathrm{Pr}_F^{M\times F}|_{\phi^* F}$, its tangent map is also just the projection $\mathrm{Pr}_{TF}^{TM\oplus TF}|_{T\phi^* F}$.

Proposition 19.3 (Bundle pullback is a contravariant functor)**.** *The map of categories*

$$\begin{aligned}
\mathit{Pullback}\colon \mathit{Manifold} &\to \left\{\mathit{Bundle}\,(M) \mid M \in \mathit{Manifold}\right\},\\
M &\mapsto \mathit{Bundle}\,(M)\,,\\
(\phi\colon M \to N) &\mapsto \Big((\mathcal{F}, F, \pi, N) \mapsto \Big(\mathcal{F}, \phi^* F, \pi^{\phi^* F}, M\Big)\Big)
\end{aligned}$$

is a contravariant functor. Here, naturally isomorphic bundles in $\mathit{Bundle}\,(M)$*, for each manifold* M*, are identified (along with the corresponding morphisms).*

Proof. Noting that

$$\mathrm{Id}_N^* F = \{(n,f) \in N \times F \mid \mathrm{Id}_N(n) = \pi(f)\} \cong F$$

and that

$$\begin{aligned}
(\mathrm{Id}_N^* \pi)(n,f) &= \left(\mathrm{Pr}_1^{N\times F}|_{\mathrm{Id}_N^* F}\right)(n,f) = n = \pi(f)\\
\implies \mathrm{Id}_N^* \pi &\cong \pi,
\end{aligned}$$

it follows that $\mathrm{Pullback}\,(\mathrm{Id}_N) = \mathrm{Id}_{\mathrm{Bundle}(N)} = \mathrm{Id}_{\mathrm{Pullback}(N)}$, i.e. Pullback satisfies the identity axiom of functoriality.

For the contravariance axiom, let $\phi\colon M \to N$ and $\psi\colon L \to M$ be smooth manifold morphisms and let $(\mathcal{F}, F, \pi, N)$ be a smooth bundle. Then

$$\begin{aligned}
\psi^*\phi^* F &= \left\{(\ell, p) \in L \times \phi^* F \mid \psi(\ell) = \pi^{\phi^* F}(p)\right\}\\
&= \left\{(\ell,(m,f)) \in L \times (M \times F) \mid \psi(\ell) = \pi^{\phi^* F}(m,f) \text{ and } \phi(m) = \pi(f)\right\}\\
&= \{(\ell,(m,f)) \in L \times (M \times F) \mid \psi(\ell) = m \text{ and } \phi(m) = \pi(f)\}\\
&\cong \{(\ell,f) \in L \times F \mid \phi \circ \psi(\ell) = \pi(f)\}\\
&= (\phi \circ \psi)^* F
\end{aligned}$$

and

$$\begin{aligned}
\pi^{\psi^*\phi^* F}(\ell,(m,f)) &= \left(\mathrm{Pr}_1^{L\times\phi^* F}|_{\psi^*\phi^* F}\right)(\ell,(m,f)) = \ell \text{ and}\\
\pi^{(\phi\circ\psi)^* F}(\ell,f) &= \left(\mathrm{Pr}_1^{L\times F}|_{(\phi\circ\psi)^* F}\right)(\ell,f) = \ell,
\end{aligned}$$

showing that $\pi^{\psi^*\phi^*F} \cong \pi^{(\phi\circ\psi)^*F}$, and therefore

$$\text{Pullback}\,(\psi) \circ \text{Pullback}\,(\phi) = \text{Pullback}\,(\phi \circ \psi)\,,$$

establishing Pullback as a contravariant functor. □

The space of sections of a pullback bundle is easily quantified.

$$\Gamma\left(\phi^*F\right) = \left\{\sigma \in C^\infty\left(M, \phi^*F\right) \mid \pi^{\phi^*F} \circ \sigma = \text{Id}_M\right\}.$$

This space will be central in the theory developed in the rest of this paper. Furthermore, it is naturally identified with the space of sections along the pullback map;

$$\Gamma_\phi\left(F\right) := \left\{\Sigma \in C^\infty\left(M, F\right) \mid \pi^F \circ \Sigma = \phi\right\}.$$

These spaces are naturally isomorphic to one another, and therefore an identification can be made when convenient. While the former space is more correct from a strongly typed standpoint, the latter space is a convenient and intuitive representational form. The particular correspondence depends heavily on the fact that ϕ^*F is a submanifold of $M \times F$.

$$\begin{aligned} \Gamma\left(\phi^*F\right) &\cong \Gamma_\phi\left(F\right) \\ \sigma &\mapsto \text{Pr}_2^{M\times F} \circ \sigma, \\ \text{Id}_M \times_M \Sigma &\leftarrow\!\shortmid \Sigma. \end{aligned}$$

Furthermore, if $f \in \Gamma\left(F\right)$, then $f \circ \phi \in \Gamma_\phi\left(F\right)$. Note that it is *not* true that any $\sigma \in \Gamma_\phi\left(F\right)$ can be written as $f \circ \phi$ for some $f \in \Gamma\left(F\right)$, for example when there exists some distinct $p, q \in M$ such that $\phi\left(p\right) = \phi\left(q\right)$ and $\sigma\left(p\right) \neq \sigma\left(q\right)$. Furthermore, the representation $f \circ \phi$ is generally non-unique, for example when ϕ is not surjective, sections $f_1, f_2 \in \Gamma\left(F\right)$ which differ only away from the image of ϕ will still give $f_1 \circ \phi = f_2 \circ \phi$. Before developing the notion of a linear connection on a pullback bundle, it will be necessary to address these features which, while inconvenient, provide the strength of the pullback bundle and pullback covariant derivative (see (21.8)).

Lemma 19.4 (Local representation of $\Gamma_\phi\left(F\right)$ elements)**.** *Recall that r denotes the rank of smooth bundle F. If $\sigma \in \Gamma_\phi\left(F\right)$ then each point $p \in M$ has some neighborhood U in which σ can be written locally as $\sigma\mid_U = \sigma^i\, f_i \circ \phi\mid_U$, where $f_1, \ldots, f_r \in \Gamma\left(F\mid_{\phi(U)}\right)$ is a frame for $F\mid_{\phi(U)}$, and $\sigma^1, \ldots, \sigma^r \in C^\infty\left(U, \mathbb{R}\right)$ are defined by $\sigma^i = \left(f^i \circ \phi\mid_U\right) \cdot_F \sigma\mid_U$.*

Proof. Let $p \in M$, let $V \subseteq N$ be a neighborhood of $\phi(p)$ over which $F|_V$ is trivial, and let $U = \phi^{-1}(V)$, so that U is a neighborhood of p. Let $f_1, \ldots, f_r \in \Gamma(F|_V)$ be a frame for $F|_V$ (i.e. $F|_{\phi(U)}$), and let $f^1, \ldots, f^r \in \Gamma((F|_V)^*)$ be the corresponding coframe (i.e. the unique $f^1, \ldots, f^r$ such that $f^i \cdot_F f_j = \delta^i_j$ for each i, j). Define $\sigma^i \in C^\infty(M, \mathbb{R})$ by $\sigma^i = \left(f^i \circ \phi|_U\right) \cdot_F \sigma|_U$. Then

$$\begin{aligned}
\sigma^i f_i \circ \phi|_U &= \left(f^i \circ \phi|_U\right) \cdot_F \sigma|_U \; f_i \circ \phi|_U \\
&= \left((f_i \circ \phi|_U) \otimes_U \left(f^i \circ \phi|_U\right)\right) \cdot_F \sigma|_U \\
&= \left(\left(f_i \otimes_V f^i\right) \circ \phi|_U\right) \cdot_F \sigma|_U \\
&= \left(\mathrm{Id}_{F|_V} \circ \phi|_U\right) \cdot_F \sigma|_U \\
&= \sigma|_U,
\end{aligned}$$

as desired. □

Some literature uses expressions of the form $f \circ \phi \in \Gamma_\phi(F)$ along with an implicit use of the section-identifying isomorphism to write down particular sections of pullback bundles. In most cases, this tacit identification of spaces is harmless, but certain highly involved calculations may suffer from it. The section that $f \circ \phi$ corresponds to under said isomorphism is $\mathrm{Id}_M \times_M (f \circ \phi) \in \Gamma(\phi^* F)$. However, because this expression is unwieldy and therefore a more compact and contextually meaningful expression is called for.

Definition 19.5 (Pullback section)**.** If $f \in \Gamma(F)$ and $\phi \colon M \to N$ is smooth, then define

$$\phi^* f := \mathrm{Id}_M \times_M (f \circ \phi) \in \Gamma(\phi^* F).$$

This is known as a **pullback section**.

The pullback section is deservedly named. If $\phi \colon M \to N$ and $\psi \colon L \to M$ are smooth, then $\psi^* \phi^* f \cong (\phi \circ \psi)^* f$ in the sense of the proof of (19.3).

Proposition 19.6 (Bundle pullback commutes with tensor product)**.** *If E and F are smooth vector bundles over manifold N and $\phi \colon M \to N$ is smooth, then the map*

$$\begin{aligned}
\phi^* E \otimes_M \phi^* F &\to \phi^*(E \otimes_N F), \\
(m, e) \otimes_M (m, f) &\mapsto (m, e \otimes_N f)
\end{aligned}$$

(extended linearly to general tensors) is a smooth vector bundle isomorphism.

Proof. Let c denote the above map. The well-definedness of c comes from the universal mapping property on multilinear forms which induces a linear map on a corresponding tensor product. If $c((m,e) \otimes_M (m,f)) = 0$, then $e \otimes_N f = 0$, which implies that $e = 0$ or $f = 0$, and therefore that $(m,e) \otimes_M (m,f) = 0$. Because there exists a basis for $(\phi^* E \otimes_M \phi^* F)_m$ consisting only of simple tensors, this implies that c is injective, and by a dimensionality argument, that c is an isomorphism. The map is clearly smooth and respects the fiber structures of its domain and codomain. Thus c is a smooth vector bundle isomorphism. □

The contravariance of pullback and its naturality with respect to tensor product are two essential properties which provide some of the flexibility and precision of the strongly typed tensor formalism described in this paper. This will become quite apparent in Part IV.

Remark 19.7 (Tensor field formulation of smooth vector bundle morphisms). A particularly useful application of pullback bundles is in forming a rich type system for smooth vector bundle morphisms. This approach was inspired by [Xin96, pg. 11]. Let $\pi^E \colon E \to M$ and $\pi^F \colon F \to N$ be smooth vector bundles, and let $\phi \colon M \to N$ be smooth. Consider $\mathrm{Hom}_\phi(E, F)$, i.e. the space of smooth vector bundle morphisms over the map ϕ. There is a natural identification with another space which lets the base map ϕ play a more direct role in the space's type. In particular,

$$\begin{aligned} \mathrm{Hom}_\phi(E,F) &\cong \mathrm{Hom}_{\mathrm{Id}_M}(E, \phi^* F), \\ A &\mapsto \pi^E \times_E A, \\ \mathrm{Pr}_2^{M \times F} \circ B &\leftarrow\!\shortmid B. \end{aligned}$$

This particular identification of smooth vector bundle morphisms over ϕ can now be directly translated into the tensor field formalism, analogously to (16.1).

$$\begin{aligned} \Gamma(\phi^* F \otimes_M E^*) &\to \mathrm{Hom}_{\mathrm{Id}_M}(E, \phi^* F), \\ A &\mapsto (e \mapsto A \cdot_E e). \end{aligned}$$

The inverse image of $B \in \mathrm{Hom}_{\mathrm{Id}_M}(E, \phi^* F)$ is given locally; let (e_i) and (f_i) denote local frames for E and F in neighborhoods $U \subseteq M$ and $V \subseteq N$ respectively, with $\phi(U) \subseteq V$, and let (e^i) and (f^i) denote their dual coframes. Then the tensor field corresponding to B is given locally in U by $B^i_j\, \phi^* f_i \otimes_M e^j$, where $B^i_j := \phi^* df^i \circ B \circ e_j \in C^\infty(U, \mathbb{R})$.

Quantifying smooth vector bundle morphisms as the tensor fields lends itself naturally to doing calculus on vector and tensor bundles, as the relevant derivatives (covariant derivatives) take the form of tensor fields. The type information for a particular vector bundle morphism is encoded in the relevant tensor bundle.

20 Tangent Map as a Tensor Field

This section deals specifically with the tangent map operator by using concepts from Section 18 and Section 19 to place it in a strongly typed setting and to prepare to unify a few seemingly disparate concepts and notation for some tangible benefit (in particular, see Section 23).

Given a smooth map $\phi\colon M \to N$, its tangent map $T\phi\colon TM \to TN$ is a smooth vector bundle morphism over ϕ, so by (19.7), is naturally identified with a tensor field

$$\mathring{\nabla}^{M \to N} \phi \in \Gamma\left(\phi^* TN \otimes_M T^* M\right),$$

which may be denoted by $\mathring{\nabla}\phi$ where type pedantry is deemed unnecessary. This construction is known as a **two-point tensor field** (see [MH83, pg. 70]). The inscribed $\circ$ symbol in $\mathring{\nabla}$ is used to denote that this is a nonlinear derivative, thereby distinguishing it from a linear covariant derivative.

Remark 20.1 (Generalized covariant derivative). The well-known one-to-one correspondence between linear connections and linear covariant derivatives (see [Lee09, pg. 520]) generalizes to a one-to-one correspondence between Ehresmann connections and a generalized notion of covariant derivative. To give a partial definition for the purposes of utility, a **generalized covariant derivative** on a smooth [fiber] bundle $F \to N$ is a map ∇ on $\Gamma(F)$ such that $\nabla\sigma \in \Gamma(\sigma^* VF \otimes_N T^* N)$ for each $\sigma \in \Gamma(F)$. The space of maps $C^\infty(M, N)$ is naturally identified as $\Gamma(N \nrightarrow M)$, and there is a natural Ehresmann connection on the bundle $N \nrightarrow M$, whose corresponding covariant derivative is the tangent map operator. This is the subject of another of the author's papers and will not be discussed here further. This is mentioned here to incorporate linear covariant derivatives (to be introduced and discussed in Section 21) and the tangent map operator (a nonlinear covariant derivative) under the single category "covariant derivative".

There is a subtle issue regarding construction of the cotangent map of ϕ which is handled easily by the tensor field construction. In particular, while the cotangent map $T^*\phi$ is the pointwise adjoint of the tangent map $T\phi$, i.e. for each $p \in M$, $T_p\phi\colon T_pM \to T_{\phi(p)}N$ is linear and $T_p^*\phi\colon T_{\phi(p)}^*N \to T_p^*M$ is the adjoint of $T_p\phi$, it does not follow that $T^*\phi \in \operatorname{Hom}(T^*N, T^*M)$, being some sort of "total adjoint" of $T\phi \in \operatorname{Hom}(TM, TN)$. The obstruction is due to the fact that ϕ may not be surjective, so there may be some fiber T_q^*N that is not of the form $T_{\phi(p)}^*N$, and therefore the domain could not be all of T^*N. Furthermore, even if ϕ were surjective, if it weren't also injective, say $\phi(p_0) = \phi(p_1)$ for some distinct $p_0, p_1 \in M$, then $T_{\phi(p_0)}^*N = T_{\phi(p_1)}^*N$, and $T_{p_0}M \neq T_{p_1}M$, so the action on the fiber $T_{\phi(p_0)}^*N$ is not well-defined.

In the tensor field parlance, the cotangent map $T^*\phi$ simply takes the form

$$(\nabla \phi)^{(1\,2)} \in \Gamma(T^*M \otimes_M \phi^*TN).$$

The permutation superscript $(1\,2)$ is used here instead of $*$ to distinguish it notationally from pullback notation, which will be necessary in later calculations. The key concept is that the tensor field $(\nabla \phi)^{(1\,2)}$ encodes the base map ϕ; the basepoint $p \in M$ is part of the domain ϕ^*T^*N itself.

The chain rule in the tensor field formalism makes use of the bundle pullback. If $\psi\colon L \to M$ is smooth, then

$$\nabla^{L\to N}(\phi \circ \psi) = \psi^* \nabla^{M\to N}\phi \cdot_{\psi^*TM} \nabla^{L\to M}\psi.$$

Because $\nabla \psi \in \Gamma(\psi^*TM \otimes_L T^*L)$, to form a well-defined natural pairing, the use of the pullback

$$\begin{aligned}\psi^* \nabla \phi &\in \Gamma(\psi^*(\phi^*TN \otimes_M T^*M)) \\ &= \Gamma(\psi^*\phi^*TN \otimes_L \psi^*T^*M) \\ &= \Gamma((\phi \circ \psi)^* TN \otimes_L \psi^*T^*M)\end{aligned}$$

is necessary (instead of just $\nabla \phi \in \Gamma(\phi^*TN \otimes_M T^*M)$).

Sometimes it is useful to discard some type information and write

$$\nabla \phi \in \Gamma_{\phi \times_M \mathrm{Id}_M} (TN \otimes_{N \times M} T^*M),$$

i.e. $\nabla \phi \colon M \to TN \otimes_{N \times M} T^*M$ such that

$$\left(\pi_N^{TN} \otimes_{N \times M} \pi_M^{T^*M}\right) \circ \nabla \phi = \phi \times_M \mathrm{Id}_M .$$

This is easily done by the canonical fiber projection available to all pullback bundle constructions; $\phi^*TN \otimes_M T^*M \cong (\phi \times_M \mathrm{Id}_M)^* (TN \otimes_{N \times M} T^*M)$, and the canonical fiber projection is

$$\rho_{TN \otimes_{N \times M} T^*M}^{(\phi \times_M \mathrm{Id}_M)^* (TN \otimes_{N \times M} T^*M)} \colon (\phi \times_M \mathrm{Id}_M)^* (TN \otimes_{N \times M} T^*M) \to TN \otimes_{N \times M} T^*M,$$

as defined in (19.2). The granularity of the type system should reflect the weight of the calculations being performed. For demonstration of contrasting situations, see the discussion at the beginning of Section 21 and the computation of the first variation in (25.1).

It is important to have notation which makes the distinction between the smooth vector bundle morphism formalism and the tensor field formalism, because it may sometimes be necessary to mix the two, though this paper will not need this. An added benefit to the tensor field formulation of tangent maps is that certain notions regarding derivatives can be conceptually and notationally combined, for example in Section 23.

21 Linear Covariant Derivatives

As will be shown in the following discussion, a linear covariant derivative (commonly referred to in the standard literature without the “linear” qualifier) provides a way to generalize the notion in elementary calculus of the differential of a vector-valued function. The linear covariant derivative interacts naturally with the notion of the pullback bundle, and this interaction leads naturally to what could be called a covariant derivative chain rule, which provides a crucial tool for the tensor calculus computations seen later.

Let V and W be finite-dimensional vector spaces let $U \subseteq V$ be open, and let $\phi \colon U \to W$ be differentiable. Recall from elementary calculus the differential $D\phi \colon U \to$

$W \otimes V^*$ (essentially matrix-valued). There is no base map information encoded in $D\phi$ (i.e. ϕ can't be recovered from $D\phi$ alone), it contains only derivative information. The vector space structure of V and W allows the trivializations $TU \cong V \nrightarrow U$ and $TW \cong W \nrightarrow W$, where the first factors are the base spaces and the second factors are the fibers (see (17.1)). The tangent map $\nabla^{U \to W}\phi \colon U \to TW \otimes_{W \times U} T^*U$ (see Section 20) has a codomain that can be trivialized similarly;

$$TW \otimes_{W \times U} T^*U \cong (W \nrightarrow W) \otimes_{W \times U} (V^* \nrightarrow U) \cong (W \otimes V^*) \nrightarrow (W \times U)\,.$$

Because $(W \otimes V^*) \nrightarrow (W \times U)$, as a set, is a direct product, it can be decomposed into two factors. Letting Pr_1 and Pr_2 be the projections onto the first and second factors respectively,

$$\mathrm{Pr}_1 \circ \nabla\phi \colon U \to W \otimes V^* \text{ and } \mathrm{Pr}_2 \circ \nabla\phi \colon U \to W \times U.$$

The map $\mathrm{Pr}_2 \circ \nabla\phi$ is the element of $\Gamma(W \nrightarrow U)$ identified with the base map ϕ itself; $\mathrm{Pr}_W^{W \times U} \circ \mathrm{Pr}_2 \circ \nabla\phi = \phi$. This base map information is discarded in defining the differential of ϕ as $D\phi := \mathrm{Pr}_1 \circ \nabla\phi$; the fiber portion of $\nabla\phi$. This construction relies critically on the natural isomorphism $TW \cong W \nrightarrow W$ for a vector space W.

An analogous construction shows that the differential $D\phi$ of a map ϕ is well-defined even when its domain is a manifold. However, when the codomain of a map ϕ is only a manifold, there does not in general exist a natural trivialization of its tangent bundle (in contrast to the vector space case), and therefore $D\phi$ can't be defined without additional structure. A linear covariant derivative provides the missing structure.

For the remainder of this section, let $\pi \colon E \to N$ define a smooth vector bundle having rank r.

A linear covariant derivative on E provides a means of taking derivatives of sections of E (i.e. maps $\sigma \colon N \to E$ such that $\pi \circ \sigma = \mathrm{Id}_N$) without passing to a higher tangent bundle as would happen under the tangent map functor (i.e. if $\sigma \in \Gamma(E)$ then $T\sigma \colon TN \to TE$ and $\nabla^{N \to E}\sigma \colon N \to TE \otimes_{E \times N} T^*N$). A linear covariant derivative provides an effective "trivialization" of TE analogous to the trivialization $TW \cong W \nrightarrow W$ as discussed above, discarding all but the "fiber" portion of the derivative, allowing the construction of an object known as the total linear covariant derivative analogous to the differential $D\phi$ as discussed above.

The notion of a linear covariant derivative on a vector bundle is arguably the crucial element of differential geometry[20]. In particular, this operator implements the product rule property common to anything that can be called a derivation – a property which is particularly conducive to the operation of tensor calculus. The total linear covariant derivative of a vector field (i.e. section of a vector bundle) allows the generalization of many constructions in elementary calculus to the setting of smooth vector bundles equipped with linear covariant derivatives. For example, the divergence $\operatorname{Div} X := \operatorname{Tr} DX$ of a vector field X on $\mathbb{R}^n$ generalizes to the divergence $\operatorname{Div} X := \operatorname{Tr} \nabla X$ of a vector field X on N, which has an analogous divergence theorem among other qualitative similarities.

Remark 21.1 (Natural linear covariant derivative on trivial line bundle)*.* Before making the general definition for the linear covariant derivative, a natural linear covariant derivative will be introduced. With N denoting a smooth manifold as before, if $f \in C^\infty(N, \mathbb{R})$, then $df \in \Gamma(T^*N)$ is the **differential** of f. Let

$$\nabla^{N \to \mathbb{R}} f := df.$$

Because $C^\infty(N, \mathbb{R})$ is naturally identified with $\Gamma(\mathbb{R} \times N)$, this is essentially the natural linear covariant derivative on the trivial line bundle $\mathbb{R} \times N$. Note that there is an associated product rule; if $f, g \in C^\infty(N, \mathbb{R})$, then $fg \in C^\infty(N, \mathbb{R})$, and

$$\nabla^{N \to \mathbb{R}}(fg) = d(fg) = g\,df + f\,dg = g \nabla^{N \to \mathbb{R}} f + f \nabla^{N \to \mathbb{R}} g.$$

When clear from context, the superscript decoration can be omitted and the derivative denoted as ∇f.

Definition 21.2 (Linear covariant derivative)**.** A **linear covariant derivative** on a vector bundle defined by $\pi \colon E \to N$ is an $\mathbb{R}$-linear map $\nabla^E \colon \Gamma(E) \to \Gamma(E \otimes_N T^*N)$ satisfying the product rule

$$\nabla^E (f \otimes_N \sigma) = \sigma \otimes_N \nabla^{N \to \mathbb{R}} f + f \otimes_N \nabla^E \sigma, \tag{21.1}$$

where $f \in C^\infty(N, \mathbb{R})$ and $\sigma \in \Gamma(E)$. The switch in order in the first term of the expression is necessary to form a tensor field of the correct type, $\Gamma(E \otimes_N T^*N)$. If

[20]The *Fundamental Lemma of Riemannian Geometry* establishes the existence of the Levi-Civita connection(see [Lee97, pg. 68]), which is a linear covariant derivative satisfying certain naturality properties.

$\sigma \in \Gamma(E)$, then the expression $\nabla^{E}\sigma$ is known as the **total [linear] covariant derivative** of σ. If $\nabla^{E}\sigma = 0$ [in a subset $U \subseteq N$], then σ is said to be **parallel** [on U]. The "linear" qualifier is implied in standard literature and is therefore often omitted.

The inscribed $\shortmid$ in ∇ is to indicate that the covariant derivative is linear, and can be omitted when clear from context, or when it is unnecessary to distinguish it from the nonlinear tangent map operator whose decorated symbol is ∇. For the remainder of this section, this distinction will not be necessary, so an undecorated ∇ will be used.

For $V \in \Gamma(TN)$, it is customary to denote $\nabla^{E}\sigma \cdot V$ by $\nabla^{E}_{V}\sigma$, where V indicates the "directional" component of the derivative. Following this convention, the product rule can be written in a form where the product rule is more obvious;

$$\nabla^{E}_{V}(f \otimes_N \sigma) = \nabla^{N \to \mathbb{R}}_{V} f \otimes_N \sigma + f \otimes_N \nabla^{E}_{V}\sigma.$$

A covariant derivative is a local operator with respect to the base space N; if $p \in N$, then $\left(\nabla^{E}\sigma\right)(p)$ depends only on the restriction of σ to an arbitrarily small neighborhood of p (see [Lee97, pg. 50]), and therefore the restriction $\nabla^{E|_U} : \Gamma_U(E) \to \Gamma_U(E \otimes_N T^*N)$ makes sense, allowing calculations using local expressions. Furthermore, a covariant derivative can be constructed locally and glued together under certain conditions. See [Lee09, pg. 503] for more on this, and as a reference for general theory on bundles, covariant derivatives, and connections.

Linear covariant derivatives on several vector bundle constructions will now be developed. In analogy to defining a linear map by its action on a generating subset (e.g. a basis or a dense subspace) and then extending using the linear structure, Lemma (21.6) allows a covariant derivative to be defined on a generating subset (which can be chosen to make the defining expression particularly natural) and then extending. In this case, the relevant space is the space of sections of the vector bundle, which is a module over the ring of smooth functions on a manifold, and the extension process is done via linearity and the product rule (see (21.2)). This approach will allow the local trivialization implementation details to be hidden within the proof of Lemma (21.6) – an example of information hiding – so that constructions of covariant derivatives can proceed clearly by focusing only on the natural properties of the relevant objects and then invoking the lemma to do the "dirty" work (see (21.7) and (21.9)).

A bit of useful notation will be introduced to simplify the next definition. If $G \subseteq \Gamma$ is a subset of a $C^{\infty}(N, \mathbb{R})$-module Γ whose elements are functions on N (and therefore have a notion of restriction to a subset) and $U \subseteq N$ is open, then let G_U denote the set of restrictions of the elements of G to the set U. Note that $G_U \subseteq \Gamma_U$ by construction.

Definition 21.3 (Finitely generating subset)**.** Say that a subset of a module **finitely generates** the module if the subset contains a finite set of generators for the module.

Definition 21.4 (Locally finitely generating subset)**.** If Γ is a $C^{\infty}(N, \mathbb{R})$-module and $G \subseteq \Gamma$, then G is said to be a **locally finitely generating subset of** Γ if each point $q \in N$ has a neighborhood $U \subseteq N$ for which G_U finitely generates Γ_U.

The space of sections of a vector bundle is the archetype for the above definition. The locally trivial nature of $\pi\colon E \to N$ allows local frames to be chosen in a neighborhood of each point of N, from which global smooth sections (though not necessarily a global frame) can be made using a partition of unity subordinate to the trivializing neighborhoods. The set of such global sections forms a locally finite generating subset of $\Gamma(E)$.

Lemma 21.5. *If G is a locally finitely generating subset of $\Gamma(E)$, then each point in N has a neighborhood $U \subseteq N$ and $e_1, \ldots, e_r \in G_U$ such that $e_1, \ldots, e_r$ forms a frame for $\Gamma_U(E)$. In other words, a local frame can be chosen out of G near each point in N.*

Proof. Let $q \in N$ and let $V \subseteq N$ be a neighborhood of q for which $G_V = \{g_1, \ldots, g_\ell\}$ finitely generates $\Gamma_V(E)$ (here, $\ell \geq r$, recalling that $r = \operatorname{Rank} E$). Without loss of generality, let $g_1(q), \ldots, g_r(q)$ be linearly independent (this is possible because the set of vectors $\{g_1(q), \ldots, g_\ell(q)\}$ spans the vector space E_q). Because g_i is continuous for each i and the linear independence of the sections $g_1, \ldots, g_r$ is an open condition (defined by $L^{-1}(\mathbb{R} \setminus \{0\})$ where $L\colon N \to \bigwedge^r E_q$, $p \mapsto g_1(p) \wedge \cdots \wedge g_r(p)$), there is a neighborhood $U \subseteq V$ of q for which $\{g_1(p), \ldots, g_r(p)\}$ is a linearly independent set for each $p \in U$. Finally, letting $e_i := g_i|_U$ for $i \in \{1, \ldots, r\}$, the sections $e_1, \ldots, e_r \in G_U$ form a frame for $\Gamma_U(E)$. ∎

The following lemma shows that defining a covariant derivative on a locally finitely generating subset of the space of sections of a vector bundle is sufficient to uniquely define a covariant derivative on the whole space. The particular generating subset can be chosen so the covariant derivative has a particularly natural expression within that subset.

Lemma 21.6 (Linear covariant derivative construction)**.** *Let G be a locally finite generating subset of $\Gamma(E)$. If $\nabla^G\colon G \to \Gamma(E \otimes_N T^*N)$ satisfies the linear covariant derivative axioms*[21]*, then there is a unique linear covariant derivative $\nabla^E\colon \Gamma(E) \to \Gamma(E \otimes_N T^*N)$ whose restriction to G is ∇^G.*

Proof. If $q \in N$, then by (21.5) there exists a neighborhood $U \subseteq N$ of q for which there are $e_1, \dots, e_r \in G_U$ forming a frame for $E\mid_U$. If $\sigma \in \Gamma(E)$, then $\sigma\mid_U = \sigma^i e_i$ for some $\sigma^1, \dots, \sigma^r \in C^\infty(U, \mathbb{R})$ (specifically, $\sigma^i = e^i \cdot_E \sigma\mid_U$, where $e^1, \dots, e^r \in \Gamma_U(E^*)$ denotes the dual coframe of $e_1, \dots, e_r$). Define $\nabla^E\colon \Gamma(E) \to \Gamma(E \otimes_N T^*N)$ locally on $\Gamma_U(E)$ so as to satisfy the product rule

$$\nabla^E(\sigma\mid_U) := e_i \otimes_N \nabla^{N\to\mathbb{R}}\sigma^i + \sigma^i \otimes_N \nabla^G e_i.$$

To show well-definedness, let $f_1, \dots, f_r \in G_U$ be another frame for $E\mid_U$. Then $\sigma = \tau^i f_i$ for some $\tau^1, \dots, \tau^r \in C^\infty(U, \mathbb{R})$. Let $\Psi\colon \Gamma_U(E) \to \Gamma_U(E)$ be the unique smooth vector bundle isomorphism such that $f_i = \Psi \cdot_E e_i$. Writing Ψ and Ψ^{-1} with respect to the frame (e_i) as $\Psi^i_j e_i \otimes e^j$ and $\left(\Psi^{-1}\right)^i_j e_i \otimes e^j$ respectively, it follows that $f_i = \Psi^j_i e_j$ and $\tau^i = \sigma^j \left(\Psi^{-1}\right)^i_j$. Then

$$\begin{aligned}
\nabla^E\left(\tau^i f_i\right) &= f_i \otimes_N \nabla^{N\to\mathbb{R}}\tau^i + \tau^i \otimes_N \nabla^G f_i \\
&= \Psi^j_i e_j \otimes_N \nabla^{N\to\mathbb{R}}\left(\sigma^k \left(\Psi^{-1}\right)^i_k\right) + \sigma^j \left(\Psi^{-1}\right)^i_j \otimes_N \nabla^G\left(\Psi^k_i e_k\right) \\
&= \Psi^j_i e_j \left(\Psi^{-1}\right)^i_k \otimes_N \nabla^{N\to\mathbb{R}}\sigma^k + \Psi^j_i e_j \sigma^k \otimes_N \nabla^{N\times\mathbb{R}}\left(\Psi^{-1}\right)^i_k \\
&\quad + \sigma^j \left(\Psi^{-1}\right)^i_j e_k \otimes_N \nabla^{N\to\mathbb{R}}\Psi^k_i + \sigma^j \left(\Psi^{-1}\right)^i_j \Psi^k_i \otimes \nabla^G e_k \\
&= \delta^j_k e_j \otimes_N \nabla^{N\to\mathbb{R}}\sigma^k + \sigma^j \delta^k_j \otimes \nabla^G e_k + \sigma^\ell e_k \otimes_N \nabla^{N\to\mathbb{R}}\left(\Psi^k_i \left(\Psi^{-1}\right)^i_\ell\right) \\
&= \nabla^E\left(\sigma^i e_i\right).
\end{aligned}$$

The last equality follows because $\Psi^k_i \left(\Psi^{-1}\right)^i_\ell = \delta^k_\ell$, which is a constant function, so $\nabla^{N\to\mathbb{R}}\left(\Psi^k_i \left(\Psi^{-1}\right)^i_\ell\right) = 0$. Thus the expression defining ∇^E doesn't depend on the choice of local frame. This establishes the well-definedness of ∇^E.

[21] What is meant by this is that the product rule must only be satisfied on $\lambda \otimes_N g$ if $\lambda g \in G$, where $\lambda \in C^\infty(N, \mathbb{R})$ and $g \in G$.

Clearly the restriction of ∇^E to G is ∇^G. This establishes the claim of existence. Uniqueness follows from the fact that ∇^E is defined in terms of the maps $\nabla^{N \to \mathbb{R}}$ and ∇^G. □

Lemma (21.6) is used in the proof of the following proposition to allow a natural formulation of the pullback covariant derivative with respect to a natural locally finite generating subset of $\Gamma(\phi^* E)$, in which the relevant derivative has a natural chain rule.

Proposition 21.7 (Pullback covariant derivative)**.** *If $\phi \colon M \to N$ is smooth and ∇^E is a covariant derivative on E, then there is a unique covariant derivative $\nabla^{\phi^* E}$ on $\phi^* E$ satisfying the chain rule*

$$\nabla^{\phi^* E} \phi^* e = \phi^* \nabla^E e \cdot_{\phi^* TN} \mathring{\nabla}^{M \to N} \phi$$

for all $e \in \Gamma(E)$.

Proof. Let $G := \{\sigma \in \Gamma(\phi^* E) \mid \sigma = \phi^* e \text{ for some } e \in \Gamma(E)\}$, noting that a local frame $e_1, \dots, e_{\operatorname{Rank} E} \in \Gamma_U(E)$ over open set $U \subseteq N$ induces a local frame $\phi^* e_1, \dots, \phi^* e_{\operatorname{Rank} E} \in \Gamma_{\phi^{-1}(U)}(\phi^* E)$, so G is a locally finite generating subset of $\Gamma(\phi^* E)$. Define

$$\begin{aligned} \nabla^G \colon G &\to \Gamma(\phi^* E \otimes_M T^* N), \\ \phi^* e &\mapsto \phi^* \nabla^E e \cdot_{\phi^* TN} \mathring{\nabla}^{M \to N} \phi. \end{aligned}$$

The well-definedness and $\mathbb{R}$-linearity of ∇^G comes from that of ∇^E. For the product rule, if $\lambda \in C^\infty(M, \mathbb{R})$ and $e \in \Gamma(E)$, then the product $\lambda \otimes_M \phi^* e$ is an element of G if and only if $\lambda = \phi^* \mu$ for some $\mu \in C^\infty(N, \mathbb{R})$, in which case, $\lambda \otimes_M \phi^* e = \phi^* \mu \otimes_M \phi^* e = \phi^*(\mu \otimes_N e)$. Then it follows that

$$\begin{aligned}
&\nabla^G (\lambda \otimes_M \phi^* e) \\
&= \nabla^G \phi^* (\mu \otimes_N e) \\
&= \phi^* \nabla^E (\mu \otimes_N e) \cdot_{\phi^* TN} \mathring{\nabla}^{M \to N} \phi \\
&= \phi^* \left(e \otimes_N \nabla^{N \to \mathbb{R}} \mu + \mu \otimes_N \nabla^E e \right) \cdot_{\phi^* TN} \mathring{\nabla}^{M \to N} \phi \\
&= \phi^* \left(e \otimes_N \nabla^{N \to \mathbb{R}} \mu \right) \cdot_{\phi^* TN} \mathring{\nabla}^{M \to N} \phi + \phi^* \left(\mu \otimes_N \nabla^E e \right) \cdot_{\phi^* TN} \mathring{\nabla}^{M \to N} \phi \\
&= \phi^* e \otimes_M \left(\phi^* \nabla^{N \to \mathbb{R}} \mu \cdot_{\phi^* TN} \mathring{\nabla}^{M \to N} \phi \right) + \phi^* \mu \otimes_M \left(\phi^* \nabla^E e \cdot_{\phi^* TN} \mathring{\nabla}^{M \to N} \phi \right) \\
&= \phi^* e \otimes_M \nabla^{M \to \mathbb{R}} \phi^* \mu + \phi^* \mu \otimes_M \nabla^G \phi^* e \\
&= \phi^* e \otimes_M \nabla^{M \to \mathbb{R}} \lambda + \lambda \otimes_M \nabla^G \phi^* e,
\end{aligned}$$

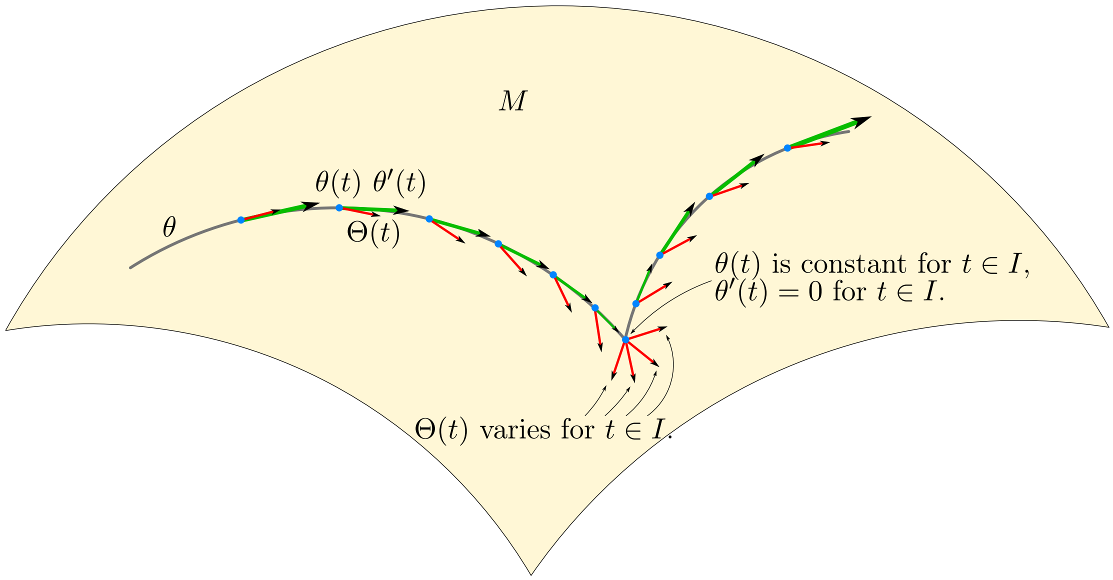


Figure 21.1: A picture of the manifold M, path θ, and vector fields θ' and Θ. The blue dots represent $\theta(t)$ at certain points $t \in \mathbb{R}$, while the green and red arrows represent $\theta'(t)$ and $\Theta(t)$ at at these points respectively. Note that Θ is a unit-length vector field along θ and varies within I, whereas θ' is a vector field along θ that vanishes within I.

which is exactly the required product rule. By (21.6), there exists a unique covariant derivative $\nabla^{\phi^* E}$ on $\phi^* E$ whose restriction to G is ∇^G. □

The full notation $\nabla^{\phi^* E}$ is often cumbersome, so it may be denoted by ∇^{ϕ} when the pulled-back bundle is clear from context.

Remark 21.8. There is an important feature of a pullback covariant derivative in the case that pullback map is not an immersion; the pullback covariant derivative may be nonzero even where the pullback map is singular. This fact can be obscured by a certain abuse of notation which often comes in the expression of the geodesic equations in differential geometry (see (25.8)). An example will illustrate this point.

Let ∇^{TM} be a covariant derivative on $\pi_M^{TM} \colon TM \to M$. Let $\Theta \colon \mathbb{R} \to TM$ be a unit-length vector field which describes the location of a person (the basepoint) and direction s/he is looking (the fiber portion) with respect to time (let $\mathbb{R}$ have standard coordinate t). Define $\theta \colon \mathbb{R} \to M$ by $\theta := \pi_M^{TM} \circ \Theta$, so that θ is the base map of Θ, i.e. θ has discarded the direction information and only encodes the location information. Say that for some closed interval $I \subseteq \mathbb{R}$, $\frac{d\theta}{dt} |_I$ is identically zero (and so is not an immersion), but that $\frac{d\Theta}{dt} |_I$ is nonvanishing; see Figure 21.1. Mathematically, this means that during this time, Θ is varying only within a single fiber of TM. Physically, this means that during this time, the person is standing still but the direction s/he is looking is changing. Passing to a higher tangent space is often undesirable (note that $\frac{d\Theta}{dt}$ takes values in TTM), so to avoid this, a covariant derivative is used. In order to be meaningful, the covariant derivative must capture this fiber-only variation.

Because Θ is a vector field along θ, it can be written as $\Theta \in \Gamma(\theta^* TM)$, and the covariant derivative on TM induces a pullback covariant derivative on $\theta^* TM$, which has base space $\mathbb{R}$. In other words, $\theta^* TM$ is parameterized by time. Then $\nabla^{\theta^* TM}_{\frac{d}{dt}} \Theta \in \Gamma(\theta^* TM)$ is the desired covariant derivative of Θ with respect to time. A coordinate-based calculation will be made to make completely obvious why this pullback covariant derivative captures the desired information. Let (x^i) be local coordinates on M and, for simplicity, assume that the image of θ lies entirely within this coordinate chart. Because (∂_i) is a local frame for TM, $(\theta^* \partial_i)$ is a local frame for $\theta^* TM$, by (19.4) and $\Theta \in \Gamma(\theta^* TM)$ can be written locally as $\Theta(t) = \Theta^i(t)\,(\theta^* \partial_i)(t)$ for some functions $\left(\Theta^i \colon \mathbb{R} \to \mathbb{R}\right)$. Then

$$\begin{aligned}
\nabla^{\theta^* TM}_{\frac{d}{dt}} \Theta &= \nabla^{\theta^* TM}_{\frac{d}{dt}} \left(\Theta^i \, \theta^* \partial_i\right) \\
&= \left(\nabla_{\frac{d}{dt}} \Theta^i\right) \theta^* \partial_i + \Theta^i \nabla^{\theta^* TM}_{\frac{d}{dt}} \theta^* \partial_i \\
&= \frac{d\Theta^i}{dt} \, \theta^* \partial_i + \Theta^i \, \theta^* \nabla^{TM} \partial_i \cdot_{\theta^* TM} \; \mathring{\nabla}^{\mathbb{R} \to M} \theta \cdot_{T\mathbb{R}} \frac{d}{dt} \\
&= \frac{d\Theta^i}{dt} \, \theta^* \partial_i + \Theta^i \, \theta^* \nabla^{TM} \partial_i \cdot_{\theta^* TM} \frac{d\theta}{dt}.
\end{aligned}$$

Note that $\mathring{\nabla}^{\mathbb{R} \to M} \theta \in \Gamma(\theta^* TM)$. Within the interval I, $\frac{d\theta}{dt}$ vanishes, so the second term vanishes on I. However, because Θ is varying in a fiber-only direction within I, the basepoint is not changing and $\frac{d\Theta^i}{dt} \theta^* \partial_i$ can be identified with an elementary vector space derivative (the fiber is a vector space and so an elementary derivative is well-defined there). This fiber-direction derivative is nonvanishing by assumption, so $\nabla^{\theta^* TM}_{\frac{d}{dt}} \Theta$ is nonvanishing on I as desired.

Introducing a bit of natural notation which will be helpful for the next result, if $X \in \Gamma(E)$ and $Y \in \Gamma(F)$, then define $X \oplus Y \equiv X \oplus_{M \times N} Y \in \Gamma(E \oplus_{M \times N} F)$ and $X \otimes Y \equiv X \otimes_{M \times N} Y \in \Gamma(E \otimes_{M \times N} F)$ by

$$(X \oplus_{M \times N} Y)(p, q) := X(p) \oplus Y(q) \qquad \text{and} \qquad (X \otimes_{M \times N} Y)(p, q) := X(p) \otimes Y(q)$$

for each $(p,q) \in M \times N$.

Proposition 21.9 (Induced covariant derivatives on $E \oplus_{M\times N} F$ and $E \otimes_{M\times N} F$)**.** *If ∇^E and ∇^F are covariant derivatives on E and F respectively, then there are unique covariant derivatives*

$$\nabla^{E\oplus_{M\times N}F} \colon \Gamma\left(E \oplus_{M\times N} F\right) \to \Gamma\left(\left(E \oplus_{M\times N} F\right) \otimes_{M\times N} \left(T^*M \oplus_{M\times N} T^*N\right)\right)$$

and

$$\nabla^{E\otimes_{M\times N}F} \colon \Gamma\left(E \otimes_{M\times N} F\right) \to \Gamma\left(\left(E \otimes_{M\times N} F\right) \otimes_{M\times N} \left(T^*M \oplus_{M\times N} T^*N\right)\right)$$

on $E \oplus F$ and $E \otimes F$ respectively, satisfying the sum rule

$$\nabla^{E\oplus F}_{u\oplus v}\left(X \oplus Y\right) = \nabla^E_u X \oplus \nabla^F_v Y$$

and the product rule

$$\nabla^{E\otimes F}_{u\oplus v}\left(X \otimes Y\right) = \nabla^E_u X \otimes Y + X \otimes \nabla^F_v Y,$$

respectively, where $X \in \Gamma(E)$, $Y \in \Gamma(F)$, and $u \oplus v \in TM \oplus TN$. Here, $TM \oplus TN \to M \times N$ (and its dual) is used instead of the isomorphic vector bundle $T(M \times N) \to M \times N$ (and its dual).

Proof. Suppressing the pedantic use of the $M \times N$ subscript to avoid unnecessary notational overload, the set $G := \{e \oplus f \mid e \in \Gamma(E),\ f \in \Gamma(F)\}$ is a locally finite generator of $\Gamma(E \oplus F)$, since local frames for $E \oplus F$ take the form $\{e_i \oplus 0, 0 \oplus f_j\}$, where $\{e_i\}$ and $\{f_j\}$ are local frames for E and F respectively. Define

$$\begin{aligned} \nabla^G \colon G &\to \Gamma\left(\left(E \oplus F\right) \otimes \left(T^*M \oplus T^*N\right)\right), \\ X \oplus Y &\mapsto \left(u \oplus v \mapsto \nabla^E_u X \oplus \nabla^F_v Y\right), \text{ where } u \oplus v \in TM \oplus TN. \end{aligned}$$

This map is well-defined and $\mathbb{R}$-linear by construction, since the connections ∇^E and ∇^F are well-defined and $\mathbb{R}$-linear. If $\lambda \in C^\infty(M \times N, \mathbb{R})$, $X \in \Gamma(E)$, and $Y \in \Gamma(F)$, then the product $\lambda \otimes (X \oplus Y)$ is in G (i.e. has the form $\overline{X} \oplus \overline{Y}$ for some $\overline{X} \in \Gamma(E)$ and $\overline{Y} \in \Gamma(F)$) if and only if λ is constant. Thus the product rule (restricted to elements of G) reduces to $\mathbb{R}$-linearity, which is already satisfied. By (21.6), there exists a unique connection $\nabla^{E\oplus F}$ on $E \oplus F$ whose restriction to G is ∇^G.

Similarly, the set $H := \{e \otimes f \mid e \in \Gamma(E), f \in \Gamma(F)\}$ is a locally finite generator of $\Gamma(E \otimes F)$, since local frames for $E \otimes F$ take the form $\{e_i \otimes f_j\}$, where $\{e_i\}$ and $\{f_j\}$ are local frames for E and F respectively. Define

$$\begin{aligned}\nabla^H \colon H &\to \Gamma\left((E \otimes F) \otimes (T^*M \oplus T^*N)\right), \\ X \otimes Y &\mapsto \left(u \oplus v \mapsto \nabla^E_u X \otimes Y + X \otimes \nabla^F_v Y\right), \text{ where } u \oplus v \in TM \oplus TN.\end{aligned}$$

This map is well-defined and $\mathbb{R}$-linear by construction, since the connections ∇^E and ∇^F are well-defined and $\mathbb{R}$-linear. For the product rule, with $\lambda \in C^\infty(M \times N, \mathbb{R})$, $X \in \Gamma(E)$, and $Y \in \Gamma(F)$, the product $\lambda \otimes (X \otimes Y)$ is in H if and only if there exist $\mu \in C^\infty(M, \mathbb{R})$ and $\nu \in C^\infty(N, \mathbb{R})$ such that $\lambda = \mu \otimes \nu \in (\mathbb{R} \nleftarrow M) \otimes (\mathbb{R} \nleftarrow N)$ (noting that then $\lambda \otimes_{M\times N} (X \otimes Y) = (\mu \otimes \nu) \otimes_{M\times N} (X \otimes Y) = (\mu \otimes_M X) \otimes (\nu \otimes_N Y)$). In this case, with $u \oplus v \in TM \oplus TN$,

$$\begin{aligned}&\nabla^H_{u\oplus v} \left(\lambda \otimes_{M\times N} (X \otimes Y)\right) \\ =\ &\nabla^H_{u\oplus v} \left((\mu \otimes \nu) \otimes_{M\times N} (X \otimes Y)\right) \\ =\ &\nabla^H_{u\oplus v} \left((\mu \otimes_M X) \otimes (\nu \otimes_N Y)\right) \\ =\ &\nabla^E_u (\mu \otimes_M X) \otimes (\nu \otimes_N Y) + (\mu \otimes_M X) \otimes \nabla^F_v (\nu \otimes_N Y) \\ =\ &\left(\nabla^{M\to\mathbb{R}}_u \mu \otimes_M X\right) \otimes (\nu \otimes_N Y) + \left(\mu \otimes_M \nabla^E_u X\right) \otimes (\nu \otimes_N Y) \\ &+ (\mu \otimes_M X) \otimes \left(\nabla^{N\to\mathbb{R}}_v \nu \otimes_N Y\right) + (\mu \otimes_M X) \otimes \left(\nu \otimes_N \nabla^F_v Y\right) \\ =\ &\left(\nabla^{M\to\mathbb{R}}_u \mu \otimes \nu + \mu \otimes \nabla^{N\to\mathbb{R}}_v \nu\right) \otimes_{M\times N} (X \otimes Y) + \lambda \otimes_{M\times N} \left(\nabla^E_u X \otimes Y + X \otimes \nabla^F_v Y\right) \\ =\ &\nabla^{M\times N\to\mathbb{R}}_{u\oplus v} \lambda \otimes_{M\times N} (X \otimes Y) + \lambda \otimes_{M\times N} \nabla^H_{u\oplus v} (X \otimes Y),\end{aligned}$$

which is exactly the required product rule. By (21.6), there exists a unique connection $\nabla^{E\otimes F}$ on $E \otimes F$ whose restriction to H is ∇^H. □

Remark 21.10 (Naturality of the covariant derivatives on $E \oplus_{M\times N} F$ and $E \otimes_{M\times N} F$). Letting $\mathrm{Pr}_i := \mathrm{Pr}^{M\times N}_i$ $(i \in \{1, 2\})$ for brevity, the maps

$$\begin{aligned}\xi \colon E \oplus_{M\times N} F &\to \mathrm{Pr}^*_1 E \oplus_{M\times N} \mathrm{Pr}^*_2 F, \\ e \oplus f &\mapsto \left(\left(\pi^E \oplus \pi^F\right)(e \oplus f), e\right) \oplus_{M\times N} \left(\left(\pi^E \oplus \pi^F\right)(e \oplus f), f\right)\end{aligned}$$

and

$$\begin{aligned}\psi \colon E \otimes_{M\times N} F &\to \mathrm{Pr}^*_1 E \otimes_{M\times N} \mathrm{Pr}^*_2 F, \\ e \otimes f &\mapsto \left(\left(\pi^E \otimes \pi^F\right)(e \otimes f), e\right) \otimes_{M\times N} \left(\left(\pi^E \otimes \pi^F\right)(e \otimes f), f\right),\end{aligned}$$

each extended linearly to the rest of their domains, are easily shown to be smooth vector bundle isomorphisms over $\mathrm{Id}_{M\times N}$. Then

$$\nabla_z^{E\oplus F}(X\oplus Y)=\xi^{-1}\left(\nabla_z^{\mathrm{Pr}_1^*E\oplus_{M\times N}\mathrm{Pr}_2^*F}\xi(X\oplus Y)\right)$$

and

$$\nabla_z^{E\otimes F}(X\otimes Y)=\psi^{-1}\left(\nabla_z^{\mathrm{Pr}_1^*E\otimes_{M\times N}\mathrm{Pr}_2^*F}\psi(X\otimes Y)\right)$$

for all $X\in\Gamma(E)$, $Y\in\Gamma(F)$, and $z\in T(M\times N)$, showing that the connections on $E\oplus F$ and $E\otimes F$ are ξ and ψ-related to the naturally induced connections on $\mathrm{Pr}_1^*E\oplus\mathrm{Pr}_2^*F$ and $\mathrm{Pr}_1^*E\otimes_{M\times N}\mathrm{Pr}_2^*F$ respectively, and are therefore in this sense natural. The sum $X\oplus Y\in\Gamma(E\oplus F)$ and product $X\otimes Y\in\Gamma(E\otimes F)$ correspond to $\mathrm{Pr}_1^*X\oplus_{M\times N}\mathrm{Pr}_2^*Y$ and $\mathrm{Pr}_1^*X\otimes_{M\times N}\mathrm{Pr}_2^*Y\in\Gamma(\mathrm{Pr}_1^*E\otimes_{M\times N}\mathrm{Pr}_2^*F)$ under ξ and ψ respectively.

Many important tensor constructions involve permutations. An extremely useful property of these permutations is that they commute with the covariant derivatives induced by the covariant derivatives on the tensor bundle factors, making them natural operators in the setting of covariant tensor calculus.

Proposition 21.11 (Transposition tensor fields are parallel)**.** *Let E_1,E_2,E_3,E_4 be smooth vector bundles over M having covariant derivatives $\nabla^{E_1},\nabla^{E_2},\nabla^{E_3},\nabla^{E_4}$ respectively, let $A:=E_1\otimes_M E_2\otimes_M E_3\otimes_M E_4$ and $B:=E_1\otimes_M E_3\otimes_M E_2\otimes_M E_4$, and let ∇^A and ∇^B denote the induced covariant derivatives.*

If $(2\,3)\in\Gamma(A^*\otimes_M B)$ denotes the tensor field which maps $e_1\otimes_M\otimes e_2\otimes_M e_3\otimes_M e_4\in A$ to $e_1\otimes_M e_3\otimes_M e_2\otimes_M e_4\in B$ (i.e. $(2\,3)$ transposes the second and third factors), then $(2\,3)$ is a parallel tensor field with respect to the covariant derivative induced on the vector bundle $A^*\otimes_M B\to M$, i.e. $\nabla^{A^*\otimes_M B}(2\,3)=0$.

Proof. Let $X \in \Gamma(TM)$. Then

$$
\begin{aligned}
& (e_1 \otimes_M e_2 \otimes_M e_3 \otimes_M e_4) \cdot_{A^*} \nabla_X^{A^* \otimes_M B} (2\,3) \\
&= \nabla_X^B \left((e_1 \otimes_M e_2 \otimes_M e_3 \otimes_M e_4) \cdot_{A^*} (2\,3)\right) - \nabla_X^A (e_1 \otimes_M e_2 \otimes_M e_3 \otimes_M e_4) \cdot_{A^*} (2\,3) \\
&= \nabla_X^B (e_1 \otimes_M e_3 \otimes_M e_3 \otimes_M e_4) \\
&\quad - \nabla_X^{E_1} e_1 \otimes_M e_3 \otimes_M e_2 \otimes_M e_4 \\
&\quad - e_1 \otimes_M \nabla_X^{E_3} e_3 \otimes_M e_2 \otimes_M e_4 \\
&\quad - e_1 \otimes_M e_3 \otimes_M \nabla_X^{E_2} e_2 \otimes_M e_4 \\
&\quad - e_1 \otimes_M e_3 \otimes_M e_2 \otimes_M \nabla_X^{E_4} e_4 \\
&= \nabla_X^B (e_1 \otimes_M e_3 \otimes_M e_3 \otimes_M e_4) - \nabla_X^B (e_1 \otimes_M e_3 \otimes_M e_3 \otimes_M e_4) \\
&= 0.
\end{aligned}
$$

Because X is arbitrary, this shows that $(e_1 \otimes_M e_2 \otimes_M e_3 \otimes_M e_4) \cdot_{A^*} \nabla^{A^* \otimes_M B} (2\,3) = 0$. This extends linearly to general tensors, so $\nabla^{A^* \otimes_M B} (2\,3) = 0$, as desired. □

The fact that all transposition tensor fields are parallel implies that all permutation tensor fields are parallel, since every permutation is just the product of transpositions. This gives as an easy corollary that a covariant derivative operation commutes with a permutation operation, which has quite a succinct statement using the permutation superscript notation.

Corollary 21.12 (Permutation tensor fields are parallel). *Let $E_1, \dots, E_k$ be smooth vector bundles over M each having a covariant derivative, and let $A := E_1 \otimes_M \cdots \otimes_M E_k$ and $B := E_{\sigma^{-1}(1)} \otimes_M \cdots \otimes_M E_{\sigma^{-1}(k)}$. If $\sigma \in S_k$ is interpreted as the tensor field in $\Gamma(A^* \otimes_M B)$ which maps $e_1 \otimes_M \cdots \otimes_M e_k$ to $e_{\sigma^{-1}(1)} \otimes_M \cdots \otimes_M e_{\sigma^{-1}(k)}$, then σ is a parallel tensor field. Stated using the superscript notation, with $X \in \Gamma(TM)$ and $a \in \Gamma(A)$,*

$$\nabla_X^B a^\sigma = \left(\nabla_X^A a\right)^\sigma .$$

Proof. This follows from the fact that σ can be written as the product of transpositions; $\nabla_X \sigma = 0$ because of the product rule and because each transposition is parallel. The claim regarding commutation with the superscript permutation follows easily from its definition.

$$\nabla_X^B a^\sigma = \nabla_X^B (a \cdot_{A^*} \sigma) = a \cdot_{A^*} \nabla_X^{A^* \otimes_M B} \sigma + \nabla_X^A a \cdot_{A^*} \sigma = \left(\nabla_X^A a\right)^\sigma ,$$

using the fact that $\nabla_X^{A^* \otimes_M B} \sigma = 0$, since σ is a parallel tensor field. □

22 Decomposition of $\pi_E^{TE} \colon TE \to E$

In using the calculus of variations on a manifold M where the Lagrangian is a function of TM (this form of Lagrangian is ubiquitous in mechanics), taking the first variation involves passing to TTM. Without a way to decompose variations into more tractable components, the standard integration-by-parts trick (see [GH96, pg. 16]) can't be applied. The notion of a local trivialization of TTM via choice of coordinates on M is one way to provide such a decomposition. A coordinate chart $(U, \phi \colon U \to \mathbb{R}^n)$ on M establishes a locally trivializing diffeomorphism $TTU \cong \phi(U) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$. However such a trivialization imposes an artificial additive structure on TTU depending on the [non-canonical] choice of coordinates, only gives a local formulation of the relevant objects, and the ensuing coordinate calculations don't give clear insight into the geometric structure of the problem. The notion of the linear connection remedies this.

A **linear connection** on the vector bundle $\pi \colon E \to M$ is a subbundle $H \to E$ of $\pi_E^{TE} \colon TE \to E$ such that $TE = H \oplus_E VE$ and $T\lambda_a \cdot H_x = H_{ax}$ for all $a \in \mathbb{R}\setminus\{0\}$ and $x \in E$, where $\lambda_a \colon E \to E$, $e \mapsto ae$ is the scalar multiplication action of a on E (see [Lee09, pg. 512]). The bundle $H \to E$ may also be called a **horizontal space** of the vector bundle $\pi_E^{TE} \colon TE \to E$ ("a" is used instead of "the" because a choice of $H \to E$ is generally non-unique). For convenience, define $h := \nabla \pi \in \Gamma(\pi^* TM \otimes_E T^*E)$, noting then that $VE = \ker h$.

A linear connection can equivalently be specified by what is known as a connection map; essentially a projection onto the vertical bundle. This is a slightly more active formulation than just the specification of a horizontal space, as a covariant derivative can be defined directly in terms of the connection map – see [Lee09, pg. 518], [EM70, pg. 128], [Eli67, pg. 173], and [Mic08, pg. 208].

Proposition 22.1 (Connection map formulation of a linear connection)**.** *If*

$$v \in \Gamma(\pi^* E \otimes_E T^*E)$$

*(i.e. $v \colon TE \to E$ is a smooth vector bundle morphism over π) is a left-inverse for $\iota_{VE}^{\pi^*E} \in \Gamma(VE \otimes_E \pi^* E^*)$ that is equivariant with respect to $T\lambda_a$ and λ_a (i.e. $v \cdot T\lambda_a = \pi^*\lambda_a \cdot v$) (see [Mic08, pg. 245]), then $H := \ker v \leq TE$ defines a linear connection on the vector bundle $\pi \colon E \to M$. Such a map v is called the **connection map** associated to H.*

Conversely, given a linear connection H, there is exactly one connection map defining H in the stated sense.

Proof. That v is a left-inverse for $\iota_{VE}^{\pi^*E}$ implies that v has full rank, so $H := \ker v$ defines a subbundle of $\pi_E^{TE} \colon TE \to E$ having the same rank as TM. Because v is smooth, H is a smooth subbundle. Furthermore, the condition implies that $V_e E \cap H_e = \{0\}$ for each $e \in E$, and therefore $TE = H \oplus_E VE$ by a rank-counting argument.

If $x \in TE$ and $a \in \mathbb{R} \setminus \{0\}$, then $v \cdot T\lambda_a \cdot x = \pi^* \lambda_a \cdot v \cdot x$, which equals zero if and only if $v \cdot x = 0$, i.e. if and only if $x \in H$. Thus $T\lambda_a \cdot H = H$. This establishes $H \to E$ as a linear connection.

Conversely, if H is a linear connection and v_1 and v_2 are connection maps for H, then $v_1 \cdot_{TE} \iota_{VE}^{\pi^*E} = \mathrm{Id}_{\pi^*E} = v_2 \cdot_{TE} \iota_{VE}^{\pi^*E}$. Then because the image of $\iota_{VE}^{\pi^*E}$ is all of VE, it follows that $v_1 \mid_{VE} = v_2 \mid_{VE}$. Since $v_1 \mid_H = 0 = v_2 \mid_H$ by definition, and since $TE = H \oplus_E VE$, this shows that $v_1 = v_2$. Uniqueness of connection maps has been established. To show existence, define $v := \iota_{\pi^*E}^{VE} \cdot_{VE} \mathrm{Pr}_{VE} \in \Gamma\left(\pi^*E \otimes_E T^*E\right)$, where $\mathrm{Pr}_{VE} \colon H \oplus_E VE \to VE$ be the canonical projection, recalling that $H \oplus_E VE = TE$. It is easily shown that v is a connection map for H. $\square$

Proposition 22.2 (Decomposing $\pi_E^{TE} \colon TE \to E$)**.** *If $v \in \Gamma\left(\pi^*E \otimes_E T^*E\right)$ is a connection map, then*

$$h \oplus_E v \colon TE \quad \to \quad \pi^*TM \oplus_E \pi^*E \tag{22.1}$$

is a smooth vector bundle isomorphism over Id_E. See Figure 22.1.

Proof. Because $TE = H \oplus_E VE$, and $H = \ker v$ and $VE = \ker h$, the fiber-wise restriction

$$h \oplus_E v \mid_{T_eE} \colon T_eE \quad \to \quad \left(\pi^*TM \oplus_E \pi^*E\right)_e \cong T_{\pi(e)}M \oplus E_{\pi(e)}$$

is a linear isomorphism for each $e \in E$. The map is a smooth vector bundle morphism over Id_E by construction. It is therefore a smooth vector bundle isomorphism over Id_E. $\square$

Remark 22.3 (Linear connection/covariant derivative correspondence)*.* Given a covariant derivative ∇^E on a smooth vector bundle $\pi \colon E \to M$, there is a naturally induced linear

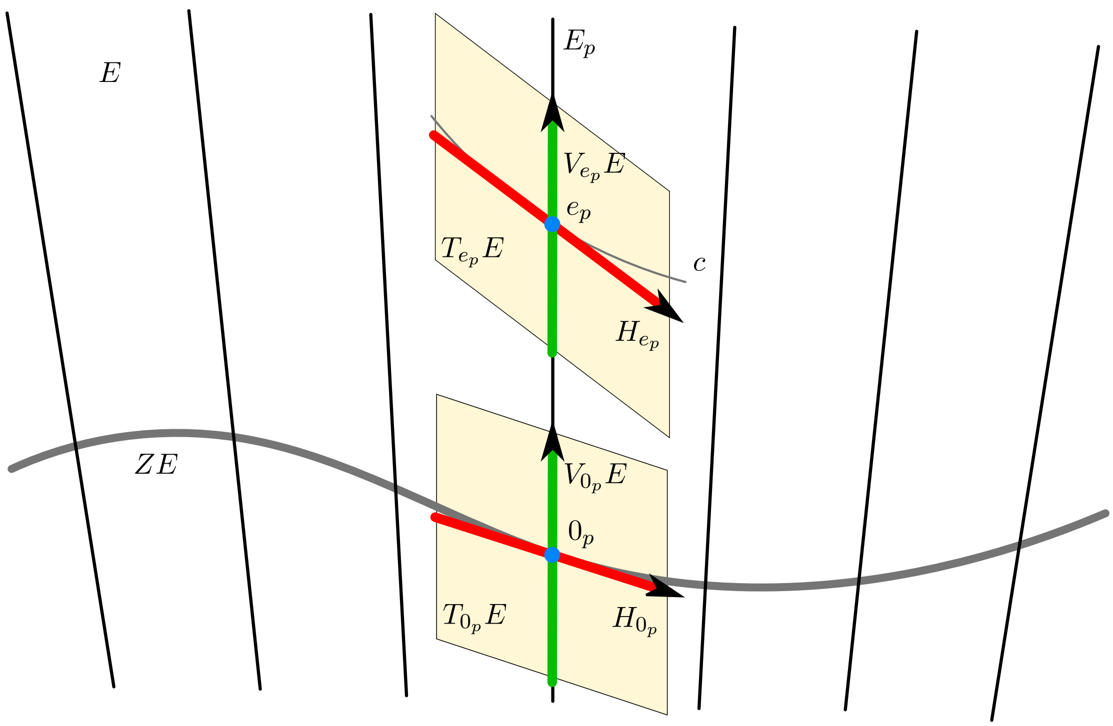


Figure 22.1: A diagram representing the decomposition of $TE \to E$ into horizontal and vertical subbundles. The vertical lines represent individual fibers of E, while $p \in M$, $e_p \in E_p$, $0_p \in E_p$ denotes the zero vector of E_p, and ZE denotes the zero subbundle of E; $ZE \cong M$. By the equivariance property of the linear connection, ZE is a submanifold of E which is entirely horizontal (its tangent space is entirely composed of horizontal vectors). The tangent spaces $T_{0_p}E$ and $T_{e_p}E$ are drawn; green arrows representing the vertical subspaces ("along" the fibers), red arrows representing the horizontal subspaces. Finally, c is a horizontal curve passing through e_p.

connection, defined via the connection map

$$\begin{aligned} v\colon TE &\to E, \\ \delta_\epsilon \Theta &\mapsto \nabla^{(\pi\circ\Theta)^* E}_{\delta_\epsilon} \Theta, \end{aligned} \tag{22.2}$$

where $\Theta\colon I \to E$ is a variation of $\theta \in E$. Here, $\nabla^{(\pi\circ\Theta)^* E}$ denotes the pullback of the covariant derivative ∇^E through the map $\pi \circ \Theta$ (see (21.7)). Conceptually, all v does is replace an ordinary derivative (δ_ϵ) with the corresponding covariant one ($\nabla^{(\pi\circ\Theta)^* E}_{\delta_\epsilon}$).

Conversely, given a connection map $v \in \Gamma(\pi^* E \otimes_E T^* E)$ for a linear connection $H \to E$, there is a naturally induced covariant derivative ∇^E on the smooth vector bundle $\pi\colon E \to M$, defined by

$$\begin{aligned} \nabla^E\colon \Gamma(E) &\to \Gamma(E \otimes_M T^* M), \\ \sigma &\mapsto \sigma^* v \cdot_{\sigma^* TE} \overline{\nabla}^{M\to E} \sigma. \end{aligned}$$

The scaling equivariance of v is critical for showing that this map actually defines a covariant derivative. Full type safety should be observed here; by the contravariance of the pullback of bundles (see (19.3)), $\sigma^* \pi^* E \cong (\pi \circ \sigma)^* E = \mathrm{Id}_M^* E \cong E$, so

$$\sigma^* v \in \Gamma(\sigma^*(\pi^* E \otimes_E T^* E)) \cong \Gamma(\sigma^* \pi^* E \otimes_M \sigma^* T^* E) \cong \Gamma(E \otimes_M \sigma^* T^* E),$$

and therefore $\sigma^* v \cdot \overline{\nabla} \sigma \in \Gamma(E \otimes_M T^* M)$ as desired. This connection map construction of a covariant derivative gives (21.7) as an immediate consequence via the chain rule for the tangent map.

The following construction is an abstraction of taking partial derivatives of a function, inspired by [MH83, pg. 277]. Instead of taking partial derivatives with respect to individual coordinates, partial covariant derivatives along distributions over the base manifold are formed, where the distributions (subbundles) decompose the base manifold's tangent bundle into a direct sum. Such a construction conveniently captures the geometry of maps with respect to the geometry of its domain.

Proposition 22.4 (Partial covariant derivatives)**.** *Let $L \in C^\infty(M, \mathbb{R})$, and for each $i \in \{1, \dots, n\}$ let $F_i \to M$ be a smooth vector bundle. If, for each $i \in \{1, \dots, n\}$, $c_i \in \Gamma(F_i \otimes_M T^* M)$ such that $c_1 \oplus_M \cdots \oplus_M c_n \in \Gamma((F_1 \oplus_M \cdots \oplus_M F_n) \otimes_M T^* M)$ is* *a smooth vector bundle isomorphism over* Id_E*, then there exist unique sections* $L_{,c_i} \in \Gamma(F_i^*)$ *for each* $i \in \{1, \ldots, n\}$ *such that*

$$\nabla^{M\to\mathbb{R}} L = L_{,c_1} \cdot_{F_1} c_1 + \cdots + L_{,c_n} \cdot_{F_n} c_n.$$

This decomposition of ∇L *provides what will be called* ***partial covariant derivatives*** *of* L *(with respect to the given decomposition).*

Proof. The following equivalences provide a formula for directly defining $L_{,c_1}, \ldots, L_{,c_n}$.

$$\nabla L = L_{,c_1} \cdot_{F_1} c_1 + \cdots + L_{,c_n} \cdot_{F_n} c_n$$

$$\iff \nabla L = (L_{,c_1} \oplus_M \cdots \oplus_M L_{,c_n}) \cdot_{F_1 \oplus_M \cdots \oplus_M F_n} (c_1 \oplus_M \cdots \oplus_M c_n)$$

$$\iff \nabla L \cdot_{TM} (c_1 \oplus_M \cdots \oplus_M c_n)^{-1} = L_{,c_1} \oplus_M \cdots \oplus_M L_{,c_n}.$$

Existence and uniqueness is therefore proven. □

Corollary 22.5 (Horizontal/vertical derivatives)**.** *Let* $h := \overline{\nabla}\, \pi \in \Gamma(\pi^* TM \otimes_E T^* E)$ *as before. If* $v \in \Gamma(\pi^* E \otimes_E T^* E)$ *is a connection map, and if* $L \colon E \to \mathbb{R}$ *is smooth, then there exist unique* $L_{,h} \in \Gamma(\pi^* T^* M)$ *and* $L_{,v} \in \Gamma(\pi^* E^*)$ *such that* $\nabla L = L_{,h} \cdot_{\pi^* TM} h + L_{,v} \cdot_{\pi^* E} v$.

It should be noted that the basepoint-preserving issue discussed in Section 20 plays a role in choosing to use the tensor field formulation of $h \colon TE \to TM$ and $v \colon TE \to E$. In particular, without preserving the basepoint (via the π-pullback of TM and E to form $h \in \Gamma(\pi^* TM \otimes_E T^* E)$ and $v \in \Gamma(\pi^* E \otimes_E T^* E)$), the map $h \oplus_E v$ would not be a smooth bundle isomorphism, and the horizontal and vertical derivatives would be maps of the form $L_{,h} \colon E \to T^* M$ and $L_{,v} \colon E \to E^*$, but that, critically, are *not* sections of smooth vector bundles, and can only claim to be smooth [fiber] bundle morphisms. Derivative trivializations will be central in calculating the first and second variations of an energy functional having Lagrangian L (see (25.1) and (26.1)).

23 Curvature and Commutation of Derivatives

A ubiquitous consideration in mathematics is to determine when two operations commute. In the setting of tensor calculus, this often manifests itself in determining the commutativity (or lack thereof) of two covariant derivatives. Here, "covariant derivatives" may refer to both linear covariant derivatives and the tangent map operator (see

(20.1)). This unified categorization of derivatives will now be leveraged to show that certain fiber bundles are flat (in a sense analogous to the vanishing of a curvature endomorphism) with respect to particular covariant derivatives. This reduces the work often done showing commutativity of derivatives in the derivation of the first variation of a function in the calculus of variations to the simple statement that a particular tensor field is symmetric, which is comes as a corollary to the aforementioned flatness.

In this section, the symbol ∇ may denote $\overset{\circ}{\nabla}$ or $\overset{|}{\nabla}$, depending on context. This eases the expression of repeated covariant derivatives, such as the covariant Hessian of a section (see below), and is an example of telescoping notation as discussed in Section 16.

If $\pi \colon E \to M$ defines a smooth [fiber] bundle whose space of sections $\Gamma(E)$ has two repeated covariant derivatives defined and if ∇^{TM} is a symmetric linear covariant derivative (meaning $\nabla_X Y - \nabla_Y X = [X, Y]$ for $X, Y \in \Gamma(TM)$), then the tensor contraction

$$\nabla^2 \sigma : (X \otimes_M Y - Y \otimes_M X)$$

is an expression measuring the non-commutativity of the X and Y derivatives of σ. The quantity $\nabla^2 \sigma$ will be called the **covariant Hessian** of σ, because it generalizes the Hessian of elementary calculus; it contains only second-derivative information, and in the special case seen below, it is symmetric in the argument components. It should be noted that if $F \to M$ is the vector bundle such that $\nabla \sigma \in \Gamma(F \otimes_M T^* M)$, then $\nabla^2 \sigma \in \Gamma(F \otimes_M T^* M \otimes_M T^* M)$. Intentionally leaving the ∇ and $\cdot$ symbols undecorated in preference of contextual interpretation, unwinding the expression above gives

$$\begin{aligned}
\nabla^2 \sigma : (X \otimes_M Y - Y \otimes_M X) &= \nabla_Y \nabla \sigma \cdot X - \nabla_X \nabla \sigma \cdot Y \\
&= \nabla_Y \nabla_X \sigma - \nabla \sigma \cdot \nabla_Y X - \nabla_X \nabla_Y \sigma + \nabla \sigma \cdot \nabla_X Y \\
&= -\nabla_X \nabla_Y \sigma + \nabla_Y \nabla_X \sigma + \nabla \sigma \cdot [X, Y] \\
&= -\nabla_X \nabla_Y \sigma + \nabla_Y \nabla_X \sigma + \nabla_{[X,Y]} \sigma,
\end{aligned}$$

which is syntactically identical to the common definition for the [Riemannian] curvature endomorphism $R(X, Y)\,\sigma$. In the traditional setting, where ∇^E is a linear covariant derivative on vector bundle E, the curvature endomorphism takes the form of a tensor field $R^E \in \Gamma(E \otimes_M E^* \otimes_M T^* M \otimes_M T^* M)$. In this setting however, because ∇^E may be nonlinear, such a tensorial formulation doesn't generally exist. Instead,

$$R^E(X,Y) := -\nabla_X \nabla_Y^E + \nabla_Y \nabla_X^E + \nabla_{[X,Y]}^E$$

defines a second-order covariant differential operator ("covariant" meaning tensorial in the X and Y components). Put differently,

$$R^E(X,Y)\,\sigma = \nabla^2\sigma : (X \otimes_M Y - Y \otimes_M X),$$

which will be called the (possibly nonlinear) **curvature operator**, measures the non-commutativity of the X and Y derivatives of σ. If R^E is identically zero, then the bundle E is said to be **flat** with respect to the relevant connections/covariant derivatives.

There are two particularly important instances of flat bundles. The first is the trivial line bundle defined by $\pi^{\mathbb{R} \twoheadrightarrow S}$ (whose space of smooth sections, as discussed in Section 17, is naturally identified with $C^\infty(S,\mathbb{R})$). In this case, $\nabla^{S\to\mathbb{R}} f \in \Gamma(T^*S)$, and $\nabla^2 f \equiv \nabla^{T^*S}\nabla^{S\to\mathbb{R}} f \in \Gamma(T^*S \otimes_S T^*S)$ is the object referred to in most literature as the covariant Hessian of f. Here, $R^{S\to\mathbb{R}}(X,Y)\,f$ is a real-valued function on S.

Proposition 23.1 (Symmetry of covariant Hessian on functions)**.** *Let S be a smooth manifold and let ∇^{TS} be a symmetric covariant derivative. If $f \in C^\infty(S,\mathbb{R})$, then $\nabla^2 f \in \Gamma(T^*S \otimes_S T^*S)$ is a symmetric tensor field (i.e. it has a $(1\,2)$ symmetry). Here, the covariant derivative on $C^\infty(S,\mathbb{R})$ is $\nabla^{S\to\mathbb{R}}$ as defined above.*

Proof. Let $X, Y \in \Gamma(TS)$. Recall that $\nabla f \equiv df \in \Gamma(T^*S)$. Then

$$\begin{aligned}
&\nabla^2 f : (X \otimes_S Y - Y \otimes_S X) \\
&= \nabla_Y \nabla f \cdot X - \nabla_X \nabla f \cdot Y \\
&= \nabla_Y (\nabla f \cdot X) - \nabla f \cdot \nabla_Y X - \nabla_X (\nabla f \cdot Y) + \nabla f \cdot \nabla_X Y \\
&= \nabla(\nabla f \cdot X) \cdot Y - \nabla(\nabla f \cdot Y) \cdot X + \nabla f \cdot [X,Y] && \text{(by symmetry of } \nabla^{TS}) \\
&= -\nabla f \cdot [X,Y] + \nabla f \cdot [X,Y] && \text{(by definition of } [X,Y]) \\
&= 0.
\end{aligned}$$

Because $X \otimes_S Y$ is pointwise-arbitrary in $TS \otimes_S TS$, this shows that $\nabla^2 f$ is symmetric. Equivalently stated, $R^{S\to\mathbb{R}}$ is identically zero, and therefore the relevant bundle is flat. □

The second important case involves the nonlinear covariant derivative $\nabla^{M\to S}$ on $C^\infty(M,S)$. Here, if $\phi \in C^\infty(M,S)$, then

$$\nabla^2\phi \equiv \nabla^{\phi^*TS\otimes_M T^*M}\nabla^{M\to S}\phi \in \Gamma\left(\phi^*TS\otimes_M T^*M\otimes_M T^*M\right),$$

so $R^{M\to S}(X,Y)\,\phi \in \Gamma(\phi^*TS)$.

Proposition 23.2 (Symmetry of covariant Hessian on maps)**.** *Let M and S be smooth manifolds and let ∇^{TM} and ∇^{TS} be symmetric covariant derivatives. If $\phi \in C^\infty(M,S)$, then $\nabla^2\phi \in \Gamma(\phi^*TS\otimes_M T^*M\otimes_M T^*M)$ is a tensor field which is symmetric in the two T^*M components (i.e. it has a $(2\,3)$ symmetry). Here, the covariant derivative on $C^\infty(M,S)$ is $\nabla^{M\to S}$ as defined above.*

Proof. Let $X,Y \in \Gamma(TM)$ and $f \in C^\infty(S,\mathbb{R})$, so that $\phi^*\nabla f \in \Gamma(\phi^*TS)$. Then

$$\begin{aligned}
&\phi^*\nabla f\cdot_{\phi^*TS} R^{M\to S}(X,Y)\,\phi\\
&= \phi^*\nabla f\cdot\left(-\nabla_X\nabla_Y\phi+\nabla_Y\nabla_X\phi+\nabla_{[X,Y]}\phi\right)\\
&= -\nabla_X\left(\phi^*\nabla f\cdot\nabla\phi\cdot Y\right)+\nabla_X\phi^*\nabla f\cdot\nabla\phi\cdot Y\\
&\quad+\nabla_Y\left(\phi^*\nabla f\cdot\nabla\phi\cdot X\right)-\nabla_Y\phi^*\nabla f\cdot\nabla\phi\cdot X\\
&\quad+\phi^*\nabla f\cdot\nabla\phi\cdot[X,Y]\\
&= -\nabla_X\left(\nabla\phi^*f\cdot Y\right)+\nabla_Y\left(\nabla\phi^*f\cdot X\right)+\nabla\phi^*f\cdot[X,Y]\\
&\quad+\left(\phi^*\nabla^2 f\cdot\nabla\phi\cdot X\right)\cdot\nabla\phi\cdot Y-\left(\phi^*\nabla^2 f\cdot\nabla\phi\cdot Y\right)\cdot\nabla\phi\cdot X\\
&= -\nabla\left(\nabla\phi^*f\cdot Y\right)\cdot X+\nabla\left(\nabla\phi^*f\cdot X\right)\cdot Y+\nabla\phi^*f\cdot[X,Y]\\
&\quad-\phi^*\nabla^2 f:\left(\left(\nabla\phi\cdot X\right)\otimes_M\left(\nabla\phi\cdot Y\right)-\left(\nabla\phi\cdot Y\right)\otimes_M\left(\nabla\phi\cdot X\right)\right).
\end{aligned}$$

By definition, $-\nabla(\nabla\phi^*f\cdot Y)\cdot X+\nabla(\nabla\phi^*f\cdot X)\cdot Y = -\nabla\phi^*f\cdot[X,Y]$, which cancels out the other term. By (23.1), $\nabla^2 f$ is symmetric, so the final term is zero. Because $\phi^*\nabla f$ is pointwise-arbitrary in ϕ^*T^*S and X and Y are pointwise-arbitrary in TM, this shows that $R^{M\to S}$ is identically zero, so the bundle defined by $\pi_M^{S\times M}\colon S\times M\to M$, whose space of sections is identified with $C^\infty(M,S)$, is flat, and therefore $\nabla^2\phi$ is symmetric in its two T^*M components. □

The construction used in (22.4) can be applied to nonlinear as well as linear covariant derivatives to considerable advantage. For example, if $\psi\colon M\times N\to L$, where M,N,L are smooth manifolds and $p_M := \mathrm{Pr}_1^{M\times N}$ and $p_N := \mathrm{Pr}_2^{M\times N}$, then define

$\psi_{,M} \in \Gamma\left(\psi^* TL \otimes_{M\times N} p_M^* T^* M\right)$ and $\psi_{,N} \in \Gamma\left(\psi^* TL \otimes_{M\times N} p_N^* T^* N\right)$ by

$$\nabla \psi = \nabla^{M\times N \to L} \psi = \psi_{,M} \cdot_{p_M^* TM} \nabla p_M + \psi_{,N} \cdot_{p_N^* TN} \nabla p_N.$$

This gives a convenient way to express partial covariant derivatives, which will be used heavily in Part IV in calculating the first and second variations of an energy functional. Note that in this parlance, $\psi_{,(M\times N)}$ is the full tangent map $\nabla \psi$.

Defining second partial covariant derivatives $\psi_{,MM}$, $\psi_{,MN}$, $\psi_{,NM}$ and $\psi_{,NN}$ by

$$\nabla\psi_{,M} = \psi_{,MM} \cdot \nabla p_M + \psi_{,MN} \cdot \nabla p_N \text{ and}$$
$$\nabla\psi_{,N} = \psi_{,NM} \cdot \nabla p_M + \psi_{,NN} \cdot \nabla p_N,$$

the symmetry of the covariant Hessian of ψ can be used to show various symmetries these second derivatives.

Proposition 23.3 (Symmetries of partial covariant derivatives)**.** *With ψ and its second partial covariant derivatives as above,*

$$\psi_{,MM} \in \Gamma\left(\psi^* TL \otimes_{M\times N} p_M^* T^* M \otimes_{M\times N} p_M^* T^* M\right)$$

and $\psi_{,NN}$ (having analogous type) are (2 3)*-symmetric (i.e.* $(\psi_{,MM})^{(2\,3)} = \psi_{,MM}$ *and* $(\psi_{,NN})^{(2\,3)} = \psi_{,NN}$*) and the mixed, second partial covariant derivatives*

$$\psi_{,MN} \in \Gamma\left(\psi^* TL \otimes_{M\times N} p_M^* T^* N \otimes_{M\times N} p_N^* T^* N\right) \text{ and}$$
$$\psi_{,NM} \in \Gamma\left(\psi^* TL \otimes_{M\times N} p_N^* T^* N \otimes_{M\times N} p_M^* T^* M\right)$$

are mutually (2 3)*-symmetric (i.e.* $\psi_{,MN} = (\psi_{,NM})^{(2\,3)}$*).*

Proof. Let $X, Y \in \Gamma(TM \oplus TN)$. If $Tp_N \cdot X = 0$ and $Tp_M \cdot Y = 0$, then

$$\begin{aligned}
0 &= \nabla^2\psi : (X \otimes_{M\times N} Y - Y \otimes_{M\times N} X) \text{ (by (23.2))} \\
&= \psi_{,MM} : (\nabla p_M \cdot X \otimes_{M\times N} \nabla p_M \cdot Y - \nabla p_M \cdot Y \otimes_{M\times N} \nabla p_M \cdot X) \\
&\quad + \psi_{,MN} : (\nabla p_M \cdot X \otimes_{M\times N} \nabla p_N \cdot Y - \nabla p_M \cdot Y \otimes_{M\times N} \nabla p_N \cdot X) \\
&\quad + \psi_{,NM} : (\nabla p_N \cdot X \otimes_{M\times N} \nabla p_M \cdot Y - \nabla p_N \cdot Y \otimes_{M\times N} \nabla p_M \cdot X) \\
&\quad + \psi_{,NN} : (\nabla p_N \cdot X \otimes_{M\times N} \nabla p_N \cdot Y - \nabla p_N \cdot Y \otimes_{M\times N} \nabla p_N \cdot X) \\
&= \psi_{,MN} : (\nabla p_M \cdot X \otimes_{M\times N} \nabla p_N \cdot Y - \nabla p_N \cdot Y \otimes_{M\times N} \nabla p_M \cdot X) \\
&= \left(\psi_{,MN} - (\psi_{,NM})^{(2\,3)}\right) : (\nabla p_M \cdot X \otimes_{M\times N} \nabla p_N \cdot Y).
\end{aligned}$$

Because $\boldsymbol{\nabla} p_M \cdot X$ and $\boldsymbol{\nabla} p_N \cdot Y$ are pointwise-arbitrary in $p_M^* TM$ and $p_N^* TN$ respectively, this implies that $\psi_{,MN} = (\psi_{,NM})^{(2\,3)}$. Analogous calculations (setting $\boldsymbol{\nabla} p_M \cdot X = 0$ and $\boldsymbol{\nabla} p_M \cdot Y = 0$ and then separately setting $\boldsymbol{\nabla} p_N \cdot X = 0$ and $\boldsymbol{\nabla} p_N \cdot Y = 0$) show that $\psi_{,MM} = (\psi_{,MM})^{(2\,3)}$ and $\psi_{,NN} = (\psi_{,NN})^{(2\,3)}$. □

There are two final results regarding the second covariant derivative that will be especially useful in the calculation of the first and second variations of an energy functional (see (25.1) and (26.2)).

Proposition 23.4 (Chain rule for covariant Hessian)**.** *Let $\pi \colon E \to N$ define a bundle having a first and second covariant derivative (i.e. a section of E can be covariantly differentiated twice). If $\phi \colon M \to N$ and $e \in \Gamma(E)$, then*

$$\nabla^2 \phi^* e = \phi^* \nabla^2 e :_{\phi^* TN} (\boldsymbol{\nabla} \phi \boxtimes_M \boldsymbol{\nabla} \phi) + \phi^* \nabla e \cdot_{\phi^* TN} \nabla \boldsymbol{\nabla} \phi.$$

Proof. Let $X \in \Gamma(TM)$. Then

$$\begin{aligned}
\nabla^2 \phi^* e \cdot X &= \nabla_X \nabla^{\phi^* E} \phi^* e \\
&= \nabla_X \left(\phi^* \nabla^E e \cdot_{\phi^* TN} \boldsymbol{\nabla} \phi \right) \\
&= \nabla_X (\phi^* \nabla e) \cdot_{\phi^* TN} \boldsymbol{\nabla} \phi + \phi^* \nabla e \cdot_{\phi^* TN} \nabla_X \boldsymbol{\nabla} \phi \\
&= \left(\phi^* \nabla^2 e \cdot_{\phi^* TN} \boldsymbol{\nabla} \phi \cdot X \right) \cdot_{\phi^* TN} \boldsymbol{\nabla} \phi + \phi^* \nabla e \cdot_{\phi^* TN} \nabla_X \boldsymbol{\nabla} \phi \\
&= \left[\phi^* \nabla^2 e :_{\phi^* TN} (\boldsymbol{\nabla} \phi \boxtimes_M \boldsymbol{\nabla} \phi) + \phi^* \nabla e \cdot_{\phi^* TN} \nabla \boldsymbol{\nabla} \phi \right] \cdot X.
\end{aligned}$$

Because X is pointwise-arbitrary in TM, this establishes the desired equality. □

Proposition 23.5 (Pullback curvature endomorphism)**.** *Let $\pi \colon E \to N$ define a vector bundle having first and second covariant derivatives. If $\phi \colon M \to N$, then $R^{\phi^* TN} = \phi^* R^{TN} :_{\phi^* TN} (\boldsymbol{\nabla} \phi \boxtimes_M \boldsymbol{\nabla} \phi)$.*

Proof. Note that

$$R^{\phi^* TN} \in \Gamma \left(\phi^* TN \otimes_M \phi^* T^* N \otimes_M T^* M \otimes_M T^* M \right).$$

Let $X, Y \in \Gamma(TM)$ and let $Z \in \Gamma(TN)$, so that $\phi^* Z \in \Gamma(\phi^* TN)$. Then

$$
\begin{aligned}
&(\mathrm{Id}_{\phi^* TN} \otimes_M \phi^* Z) \cdot_{\phi^* TN \otimes_M \phi^* T^* N} R^{\phi^* TN} :_{TM} (X \otimes_M Y) \\
&= R^{\phi^* TN}(X, Y)(\phi^* Z) \\
&= \nabla^2 \phi^* Z :_{TM} (X \wedge_M Y) \\
&= \phi^* \nabla^2 Z :_{\phi^* TN} (\mathring{\nabla} \phi \boxtimes_M \mathring{\nabla} \phi) :_{TM} (X \wedge_M Y) \\
&\quad + \phi^* \nabla Z \cdot_{\phi^* TN} \nabla \mathring{\nabla} \phi :_{TM} (X \wedge_M Y) \\
&\quad \text{(by (23.4))} \\
&= \phi^* \nabla^2 Z :_{\phi^* TN} ((\mathring{\nabla} \phi \cdot X) \wedge_M (\mathring{\nabla} \phi \cdot Y)) + \phi^* \nabla Z \cdot_{\phi^* TN} 0 \\
&\quad \text{(by symmetry of } \nabla \mathring{\nabla} \phi \text{)} \\
&= \left(\phi^* \left((\mathrm{Id}_{TN} \otimes_N Z) \cdot_{TN \otimes_N T^* N} R^{TN} \right) \right) :_{\phi^* TN} ((\mathring{\nabla} \phi \cdot X) \otimes_M (\mathring{\nabla} \phi \cdot Y)) \\
&= (\mathrm{Id}_{\phi^* TN} \otimes_M \phi^* Z) \cdot_{\phi^* TN \otimes_M \phi^* T^* N} \phi^* R^{TN} :_{\phi^* TN} (\mathring{\nabla} \phi \boxtimes_M \mathring{\nabla} \phi) :_{TM} (X \otimes_M Y),
\end{aligned}
$$

and because X, Y and $\phi^* Z$ are pointwise-arbitrary in their respective spaces, this establishes the desired equality. □

A common operation is to evaluate a covariant derivative along a single tangent vector. One can express a single tangent vector as a section of a particular pullback bundle, the map being the constant map evaluating to the basepoint of the vector. This allows the richly-typed formalism of pullback bundles to be used to evaluate derivatives at a point, particularly noting that this safely deals with the overloading of the natural pairing operator $\cdot$ (see Section 18).

Proposition 23.6 (Evaluation commutes with non-involved derivatives)**.** *Let A and B be smooth manifolds and let $\sigma \in \Gamma(E)$ for some smooth bundle $E \to A \times B$ having a covariant derivative ∇^E. If $b \in B$ and the map $z \colon A \to A \times B$, $a \mapsto (a, b)$ represents evaluation at b, then*

$$
z^* (\sigma_{,A}) = (z^* \sigma)_{,A},
$$

i.e. evaluation in B commutes with a derivative along A.

Proof. Let $X \in \Gamma(TA)$, and let $p_A := \mathrm{Pr}_1^{A \times B}$ and $p_B := \mathrm{Pr}_2^{A \times B}$. Then

$$\begin{aligned}
(z^*\sigma)_{,A} \cdot X &= \nabla^{z^*E} z^*\sigma \cdot X \\
&= z^*\nabla^E \sigma \cdot_{z^*(TA \oplus TB)} \mathring{\nabla}\, z \cdot_{TA} X \\
&= z^* \left(\nabla^E \sigma \cdot_{TA \oplus TB} p_A^* X\right) \\
&= z^* \left(\sigma_{,A} \cdot_{p_A^* TA} \mathring{\nabla}\, p_A \cdot_{TA \oplus TB} p_A^* X\right) \\
&\text{(since } \mathring{\nabla}\, p_B \cdot_{TA \oplus TB} p_A^* X = 0) \\
&= z^*\sigma_{,A} \cdot_{TA} X \\
&\text{(since } z^* p_A^* X = (p_A \circ z)^* X = \mathrm{Id}_A^* X = X),
\end{aligned}$$

and because X is pointwise-arbitrary in TM, this implies that $z^*\sigma_{,A} = (z^*\sigma)_{,A}$ as desired. □

Proposition 23.7. *Let A, B, C be smooth manifolds, let $\psi \colon A \times B \to C$ be smooth, let $p_A := \mathrm{Pr}_1^{A \times B}$ and $p_B := \mathrm{Pr}_2^{A \times B}$, and let $X, Y \in \Gamma(TA \oplus TB)$. If $\mathring{\nabla}\, p_B \cdot X = 0$ and $\mathring{\nabla}\, p_A \cdot Y = 0$, then*

$$\psi_{,AB} : \left((\mathring{\nabla}\, p_A \cdot X) \otimes_{A \times B} (\mathring{\nabla}\, p_B \cdot Y)\right) = \nabla_Y^{\psi^* TC} \nabla_X^{A \times B \to C} \psi.$$

Proof. The conditions $\mathring{\nabla}\, p_B \cdot X = 0$ and $\mathring{\nabla}\, p_A \cdot Y = 0$ imply that $\nabla_Y X = 0$ in the product covariant derivative. Then since $p_A \times_{A \times B} p_B = \mathrm{Id}_{A \times B}$, it follows that

$$\nabla \mathring{\nabla}\, p_A \oplus_{A \times B} \nabla \mathring{\nabla}\, p_B = \nabla \left(\mathring{\nabla}\, p_A \oplus_{A \times B} \mathring{\nabla}\, p_B\right) = \nabla \mathring{\nabla} \left(p_A \times_{A \times B} p_B\right) = \nabla \mathrm{Id}_{TA \oplus TB} = 0,$$

and therefore

$$\nabla_Y \left(\mathring{\nabla}\, p_A \cdot X\right) = \nabla_Y \mathring{\nabla}\, p_A \cdot X + \mathring{\nabla}\, p_A \cdot \nabla_Y X = 0 \cdot X + \mathring{\nabla}\, p_A \cdot 0 = 0.$$

For the main calculation,

$$
\begin{aligned}
&\psi_{,AB} : ((\boldsymbol{\nabla} p_A \cdot X) \otimes_{A\times B} (\boldsymbol{\nabla} p_B \cdot Y)) \\
=\ &\left(\psi_{,AB} \cdot_{p_B^* TB} \boldsymbol{\nabla} p_B \cdot_{TA\oplus TB} Y\right) \cdot_{p_A^* TA} \boldsymbol{\nabla} p_A \cdot_{TA\oplus TB} X \\
=\ &\left(\nabla_Y^{\psi\times_{A\times B} p_A} \psi_{,A}\right) \cdot_{p_A^* TA} \boldsymbol{\nabla} p_A \cdot_{TA\oplus TB} X \\
&(\text{since } \boldsymbol{\nabla} p_A \cdot Y = 0) \\
=\ &\nabla_Y^{\psi} (\psi_{,A} \cdot \boldsymbol{\nabla} p_A \cdot X) - \psi_{,A} \cdot \nabla_Y^{p_A} (\boldsymbol{\nabla} p_A \cdot X) \\
&(\text{by reverse product rule}) \\
=\ &\nabla_Y^{\psi^* TC} \nabla_X^{A\times B \to C} \psi \\
&(\text{since } \nabla_Y (\boldsymbol{\nabla} p_A \cdot X) = 0),
\end{aligned}
$$

as desired. □

Part IV

Riemannian Calculus of Variations

The use of the Calculus of Variations in the Riemannian setting to develop the geodesic equations and to study harmonic maps is quite well-established. A more general formulation is required for more specific applications, such as continuum mechanics in Riemannian manifolds. The tools developed in Part III will now be used to formulate the first and second variations and Euler-Lagrange equations of an energy functional corresponding to a first-order Lagrangian. In particular, the bundle decomposition discussed in Section 22 will be needed to employ the standard integration-by-parts trick seen in the formulation of the analogous parts of the elementary Calculus of Variations. The seemingly heavy and pedantic formalism built up thus far will now show its usefulness.

In this part, let (M, g) and (S, h) be Riemannian manifolds with M compact. Calculations will be done formally in the space $C^\infty(M, S)$, noting that its completion under various norms will give various Sobolev spaces of maps from M to S, which are ultimately the spaces which must be considered when finding critical points of the relevant energy functionals. See [EM70, Eli67] for details on the analytical issues. Let dV_g denote the Riemannian volume form corresponding to metric g, and let $d\overline{V}_g$ be the induced volume form on ∂M. Let $\iota\colon \partial M \to M$ be the inclusion, and let $\nu \in \Gamma(\iota^* T^* M)$ be the unit normal covector field on ∂M. Let $E := TS \otimes_{S\times M} T^*M$ and $\pi := \pi_S^{TS} \otimes_{S\times M} \pi_M^{T^*M}$, making $\pi\colon E \to S \times M$ a vector bundle.

The energy functionals in this section will be assumed to have the form

$$\begin{aligned} \mathcal{L}\colon C^\infty(M, S) &\to \mathbb{R}, \\ \phi &\mapsto \int_M L \circ \boldsymbol{\nabla} \phi \, dV_g, \end{aligned}$$

where $L\colon E \to \mathbb{R}$, referred to as the **Lagrangian** of the functional, is smooth. Here, $\boldsymbol{\nabla} \phi$ could be understood to take values either in $E = TS \otimes_{S\times M} T^*M$ or $\phi^* TS \otimes_M T^*M$. In the former case, the composition $L \circ \boldsymbol{\nabla} \phi$ is literal, while in the latter case, there is an implicit conversion from $\phi^* TS \otimes_M T^*M$ to $TS \otimes_{S\times M} T^*M$ via a fiber projection bundle morphism (see (19.2)). Either way, $L \circ \boldsymbol{\nabla} \phi\colon M \to \mathbb{R}$. Let ∇^{TS} and ∇^{TM} denote the respective Levi-Civita connections, which induce a covariant derivative ∇^E on E (see (21.9)). Define the connection map $v \in \Gamma(\pi^* E \otimes_E T^* E)$ using ∇^E as in (22.2). For convenience, the $S \times M$ subscript will be suppressed on the "full" tensor product defining E from here forward.

24 Critical Points and Variations

One of the most pertinent properties of an energy functional is its set of critical points. Often, the solution to a problem in physics will take the form of minimizing a particular energy functional. Lagrangian mechanics is the quintessential example of this. This section will deal with some of the main considerations regarding such critical points.

Because the domain of a [real-valued] functional $\mathcal{L}$ may be a nonlinear space, the relevant first derivative is the [real-valued] differential $d\mathcal{L}$, which is paired with the linearized variation of a map $\phi \in C^\infty(M, S)$. In particular, a **one-parameter variation** of ϕ is a smooth map $\Phi \colon M \times I \to S$, where the I component is the variational parameter. Letting i denote the standard coordinate on I, the linearized variation is then $\delta_i\Phi \colon M \to TS$, recalling that $\delta_i := \frac{\partial}{\partial i} \mid_{i=0}$. Because $\pi_S^{TS} \circ \delta_i\Phi = \phi$, it follows that $\delta_i\Phi \in \Gamma(\phi^* TS)$, i.e. $\delta_i\Phi$ is a vector field along ϕ. The object $\delta_i\Phi$ will be called a **linearized variation**. Call the elements of $\Gamma(\phi^* TS)$ **linear variations**.

Proposition 24.1 (Each linear variation is a linearized variation)**.** *Let* $\exp \colon U \to S$ *denote the exponential map associated to* ∇^{TS}*, where* $U \subseteq TS$ *is a neighborhood of the zero bundle in* TS *on which* $\exp$ *is defined, and let* $\lambda \colon TS \times \mathbb{R} \to TS$*,* $(s, \epsilon) \mapsto \epsilon s$ *denote the scalar multiplication structure on* TS*. If* $A \in \Gamma(\phi^* TS)$ *and if* $\Phi \colon U \to S$ *is defined by* $\Phi := \exp \circ \lambda \circ (A \times \mathrm{Id}_I) \mid_U$*, then* $\delta_i\Phi = A$*. In other words, every vector field over* ϕ *is realized as the linearization of a one-parameter variation of* ϕ*.*

Proof. The map Φ is well-defined and smooth by construction. Let $p \in M$. Then

$$\begin{aligned}
(\delta_i\Phi)(p) &= \delta_i(\Phi(p, i)) \\
&= \delta_i(\exp \circ \lambda \circ (A \times \mathrm{Id}_I)(p, i)) \\
&= \nabla \exp \cdot \delta_i(\lambda(A(p), i)) \\
&= \nabla \exp \cdot \delta_i(iA(p)) \\
&= \nabla \exp \cdot \left(\iota_{VE}^{\pi^* E} \mid_{Z(\pi^* E)}\right) \cdot A(p) \\
&= A(p),
\end{aligned}$$

where $Z(\pi^* E)$ denotes the zero subbundle of $\pi^* E$. The last equality follows from a naturality property of the exponential map (see [Lee06, pg. 523]). □

Thus each linear variation is a linearized variation, establishing a natural identification of $T_\phi\left(C^\infty\left(M,S\right)\right)$ with $\Gamma\left(\phi^* TS\right)$, which will be useful when calculating the differential of a functional on $C^\infty\left(M,S\right)$. In fact, the exponential map construction in (24.1) is a way to construct charts for the infinite dimensional manifold $C^\infty\left(M,S\right)$ (see [Eli67, Theorem 5.2]).

25 First Variation

This section is devoted to calculating the first variation of the previously defined energy functional. Here is where the full richness of the type system of the objects developed earlier in the paper will really show their power (and arguably, necessity). While the type-specifying notation may appear overly decorated and pedantic, subtle usage errors can be detected and avoided by keeping track of the myriad of types of the relevant objects through the sub/superscripts on covariant derivatives and natural pairings; extremely complex constructions can be made and navigated without much trouble. By contrast, performing the ensuing calculations in coordinate trivializations would result in an intractable proliferation of Christoffel symbols and indexed expressions which would prove difficult to read and would be highly prone to error.

Because the Lagrangian $L\colon E \to \mathbb{R}$ is defined on a vector bundle $\pi\colon E \to S \times M$ over the product space $S \times M$, the decomposition in (22.5) can be slightly refined. The projection π can be decomposed into the factors $\pi_S := \mathrm{Pr}_S^{S\times M} \circ \pi$ and $\pi_M := \mathrm{Pr}_M^{S\times M} \circ \pi$, so that $\pi = \pi_S \times_E \pi_M$. Then $h = \overline{\nabla}\,\pi = \overline{\nabla}\,\pi_S \oplus_E \overline{\nabla}\,\pi_M$. Let

$$\sigma := \overline{\nabla}\,\pi_S \in \Gamma\left(\pi_S^* TS \otimes_E T^* E\right) \text{ and } \mu := \overline{\nabla}\,\pi_M \in \Gamma\left(\pi_M^* TM \otimes_E T^* E\right).$$

The letters *sigma* and *mu* have been chosen to reflect the fact that $L_{,\sigma} \in \Gamma\left(\pi_S^* T^* S\right)$ and $L_{,\mu} \in \Gamma\left(\pi_M^* T^* M\right)$ give the "S component" (spatial) and "M component" (material) of the derivative $\nabla^{E\to\mathbb{R}} L \in \Gamma\left(T^* E\right)$. The connection map v will be retained as is, giving $L_{,v} \in \Gamma\left(\pi^* E^*\right)$, the "$E$ component" (fiber) of $\nabla^{E\to\mathbb{R}} L$. See (25.5) for a discussion of how the quantities $L_{,\mu}, L_{,\sigma}, L_{,v}$ generalize the analogous structures in the elementary treatment of the calculus of variations.

Because a one-parameter variation of $\phi \in C^\infty\left(M,S\right)$ has the form $\Phi\colon M \times I \to S$ but the energy functional $\mathcal{L}$ involves only the M derivative of its argument, the partial tangent map must be used here. For the purposes of calculating the first and second variations, $\mathcal{L}$ must be written as

$$\mathcal{L}(\phi) := \int_M L \circ \phi_{,M} \, dV_g.$$

Theorem 25.1 (First variation of $\mathcal{L}$)**.** *Let $\mathcal{L}$, L, σ, μ, v and ν all be defined as above. If $\phi \in C^\infty(M, S)$ and $A \in \Gamma(\phi^* TS)$, then*

$$d\mathcal{L}(\phi) \cdot A = \int_M A \cdot_{\phi^* T^* S} \left(\phi_{,M}^* L_{,\sigma} - \operatorname{Div}_M \left(\phi_{,M}^* L_{,v}\right)\right) dV_g + \int_{\partial M} A \cdot_{\phi^* T^* S} \phi_{,M}^* L_{,v} \cdot_{T^* M} \nu \, d\overline{V}_g.$$

The expression above is often called the **first variation** of $\mathcal{L}$. A type analysis here gives $\phi_{,M}^* L_{,\sigma} \in \Gamma(\phi^* T^* S)$ and $\phi_{,M}^* L_{,v} \in \Gamma(\phi^* T^* S \otimes_M TM)$. Recall that because the domain of ϕ is M, $\nabla \phi \equiv \phi_{,M}$.

Proof. Supporting calculations will be made below in lemmas. Let $\Phi \colon M \times I \to S$ be as in (24.1), so that $\delta_i \Phi = A$. For tidiness, let $\mathbf{L}_{,\sigma} := \phi_{,M}^* L_{,\sigma}$ and $\mathbf{L}_{,v} := \phi_{,M}^* L_{,v}$. Then

$$\begin{aligned}
d\mathcal{L}(\phi) \cdot A &= d\mathcal{L}(\phi) \cdot \delta_i \Phi \\
&= \delta_i (\mathcal{L}(\Phi)) \\
&= \int_M \delta_i (L \circ \Phi_{,M}) \, dV_g \\
&= \int_M \mathbf{L}_{,\sigma} \cdot_{\phi^* TS} A + \mathbf{L}_{,v} \cdot_{\phi^* TS \otimes_M T^* M} \nabla^{\phi^* TS} A \, dV_g \\
&\quad \text{(by (25.2))} \\
&= \int_M A \cdot_{\phi^* T^* S} (\mathbf{L}_{,\sigma} - \operatorname{Div}_M \mathbf{L}_{,V}) + \operatorname{Div}_M (A \cdot_{\phi^* T^* S} \mathbf{L}_{,V}) \, dV_g \\
&\quad \text{(by (25.2))} \\
&= \int_M A \cdot_{\phi^* T^* S} (\mathbf{L}_{,\sigma} - \operatorname{Div}_M \mathbf{L}_{,V}) \, dV_g + \int_{\partial M} A \cdot_{\phi^* T^* S} \mathbf{L}_{,V} \cdot_{T^* M} \nu \, d\overline{V}_g \\
&\quad \text{(divergence theorem)},
\end{aligned}$$

as desired.

As for the types of $\phi_{,M}^* L_{,\sigma}$ and $\phi_{,M}^* L_{,v}$, the contravariance of bundle pullback allows significant simplification. Because $L_{,\sigma} \in \Gamma(\pi_S^* T^* S)$ and $L_{,v} \in \Gamma(\pi^* E^*)$,

$$\begin{aligned}
&\phi_{,M}^* L_{,\sigma} \in \Gamma\left(\phi_{,M}^* \pi_S^* T^* S\right) = \Gamma\left((\pi_S \circ \phi_{,M})^* T^* S\right) = \Gamma(\phi^* T^* S) \text{ and} \\
&\phi_{,M}^* L_{,v} \in \Gamma\left(\phi_{,M}^* \pi^* E\right) = \Gamma\left((\pi \circ \phi_{,M})^* (T^* S \otimes TM)\right) = \Gamma(\phi^* T^* S \otimes_M TM).
\end{aligned}$$

The supporting calculations follow. Define $z\colon M \to M \times I$, $m \mapsto (m,0)$ for purposes of evaluation of $i = 0$ via precomposition as in (23.6). Then δ_i is a section of a pullback bundle; $\delta_i = z^*\partial_i \in \Gamma\left(z^*\left(TM \oplus TI\right)\right)$. It should be noted that $\Phi \circ z = \phi$ by definition, and that $z^*\Phi_{,M} = \left(z^*\Phi\right)_{,M} = \phi_{,M}$ by (23.6). $\square$

Lemma 25.2. *Let L, Φ, A, σ, and v be as in Theorem 25.1. The variational derivative of $L \circ \Phi_{,M}$ decomposes in terms of the partial covariant derivatives $\mathbf{L}_{,\sigma}$ and $\mathbf{L}_{,v}$ and the linearized variation A;*

$$\delta_i \left(L \circ \Phi_{,M}\right) = \phi_{,M}^* L_{,\sigma} \cdot_{\phi^*TS} \delta_i\Phi + \phi_{,M}^* L_{,v} \cdot_{(\phi \times_M \mathrm{Id}_M)^*E} \nabla^{\phi}\delta_i\Phi.$$

The integration-by-parts trick as in the derivation of the first variation in elementary calculus of variations generalizes to the covariant setting;

$$\mathbf{L}_{,\sigma} \cdot_{\phi^*TS} A + \mathbf{L}_{,v} \cdot_{\phi^*TS \otimes_M T^*M} \nabla^{\phi} A = A \cdot_{\phi^*T^*S} \left(\mathbf{L}_{,\sigma} - \mathrm{Div}_M \mathbf{L}_{,v}\right) + \mathrm{Div}_M \left(A \cdot_{\phi^*T^*S} \mathbf{L}_{,v}\right).$$

Proof. A wonderful string of equalities follows.

$$\begin{aligned}
&\delta_i \left(L \circ \Phi_{,M}\right) \\
&= z^*\nabla^{M\times I \to \mathbb{R}} \left(L \circ \Phi_{,M}\right) \cdot_{z^*(TM \oplus TI)} \delta_i \\
&\quad \text{(here, } \delta_i = z^*\partial_i\text{)} \\
&= z^*\Phi_{,M}^* \nabla^{E \to \mathbb{R}} L \cdot_{z^*\Phi_{,M}^* TE} z^* \nabla^{M \times I \to E} \Phi_{,M} \cdot_{z^*(TM\oplus TI)} \delta_i \\
&\quad \text{(chain rule)} \\
&= \phi_{,M}^* \left(L_{,\sigma} \cdot_{\pi_S^* TS} \sigma + L_{,\mu} \cdot_{\pi_M^* TM} \mu + L_{,v} \cdot_{\pi^* E} v\right) \cdot_{\phi_{,M}^* TE} \delta_i \Phi_{,M} \\
&\quad \text{(by (22.4) and because } \Phi_{,M} \circ z = \phi_{,M}\text{)} \\
&= \phi_{,M}^* L_{,\sigma} \cdot_{\phi_{,M}^* \pi_S^* TS} \phi_{,M}^* \sigma \cdot_{\phi_{,M}^* TE} \delta_i \Phi_{,M} \\
&\quad + \phi_{,M}^* L_{,\mu} \cdot_{\phi_{,M}^* \pi_M^* TM} \phi_{,M}^* \mu \cdot_{\phi_{,M}^* TE} \delta_i \Phi_{,M} \\
&\quad + \phi_{,M}^* L_{,v} \cdot_{\phi_{,M}^* \pi^* E} \phi_{,M}^* v \cdot_{\phi_{,M}^* TE} \delta_i \Phi_{,M} \\
&= \phi_{,M}^* L_{,\sigma} \cdot_{\phi^* TS} \delta_i \Phi + \phi_{,M}^* L_{,v} \cdot_{(\phi \times_M \mathrm{Id}_M)^* E} \nabla^{\phi} \delta_i \Phi \\
&\quad \text{(by (25.3)).}
\end{aligned}$$

Note that by (19.3),

$$\Phi_{,M}^{*}\pi_S^{*}TS = (\pi_S \circ \Phi_{,M})^{*}\, TS = \Phi^{*}TS,$$
$$\Phi_{,M}^{*}\pi_M^{*}TM = (\pi_M \circ \Phi_{,M})^{*}\, TM = \left(\mathrm{Pr}_M^{M\times I}\right)^{*} TM, \text{ and}$$
$$\Phi_{,M}^{*}\pi^{*}E = (\pi \circ \Phi_{,M})^{*}\, E = \left(\Phi \times_{M\times I} \mathrm{Pr}_M^{M\times I}\right)^{*} E.$$

Replacing $\delta_i\Phi$ with A gives

$$\delta_i\left(L \circ \Phi_{,M}\right) = \mathbf{L}_{,\sigma} \cdot_{\phi^{*}TS} A + \mathbf{L}_{,v} \cdot_{\phi^{*}TS\otimes_M T^{*}M} \nabla^{\phi} A,$$

establishing the first equality.

For the second,

$$\begin{aligned}
&\mathbf{L}_{,\sigma} \cdot_{\phi^{*}TS} A + \mathbf{L}_{,v} \cdot_{\phi^{*}TS\otimes_M T^{*}M} \nabla^{\phi} A\\
&= \mathbf{L}_{,\sigma} \cdot_{\phi^{*}TS} A + \mathrm{Tr}_{T^{*}M}\left(\mathbf{L}_{,v} \cdot_{\phi^{*}TS} \nabla^{\phi} A\right)\\
&\quad \text{(tracing } TM \text{ separately)}\\
&= A \cdot_{\phi^{*}T^{*}S} \mathbf{L}_{,\sigma} + \mathrm{Tr}_{T^{*}M}\left(\nabla\left(\mathbf{L}_{,v} \cdot_{\phi^{*}TS} A\right) - \left(\nabla^{\phi\times_M \mathrm{Id}_M}\mathbf{L}_{,v}\right) \cdot_{\phi^{*}TS} A\right)\\
&\quad \text{(reverse product rule)}\\
&= A \cdot_{\phi^{*}T^{*}S} \mathbf{L}_{,\sigma} - A \cdot_{\phi^{*}T^{*}S} \mathrm{Tr}_{T^{*}M} \nabla^{\phi\times_M \mathrm{Id}_M}\mathbf{L}_{,v} + \mathrm{Tr}_{T^{*}M}\nabla\left(A \cdot_{\phi^{*}T^{*}S} \mathbf{L}_{,v}\right)\\
&\quad (\cdot_{\phi^{*}TS}\text{commutes with } \mathrm{Tr}_{T^{*}M})\\
&= A \cdot_{\phi^{*}T^{*}S} \left(\mathbf{L}_{,\sigma} - \mathrm{Div}_M \mathbf{L}_{,v}\right) + \mathrm{Div}_M\left(A \cdot_{\phi^{*}T^{*}S} \mathbf{L}_{,v}\right)\\
&\quad \text{(definition of } \mathrm{Div}_M\text{)}.
\end{aligned}$$

Note that $\mathbf{L}_{,v} \in \Gamma\left(\phi^{*}T^{*}S \otimes_M TM\right)$, so $\mathrm{Div}_M \mathbf{L}_{,v} \in \Gamma\left(\phi^{*}T^{*}S\right)$ and $A \cdot_{\phi^{*}T^{*}S} \mathbf{L}_{,v} \in \Gamma\left(TM\right)$. □

Lemma 25.3. *The variation $\delta_i\Phi_{,M}$ decomposes as follows.*

$$\phi_{,M}^{*}\sigma \cdot_{\phi_{,M}^{*}TE} \delta_i\Phi_{,M} = \delta_i\Phi \in \Gamma\left(\phi^{*}TS\right),$$
$$\phi_{,M}^{*}\mu \cdot_{\phi_{,M}^{*}TE} \delta_i\Phi_{,M} = 0 \in \Gamma\left(TM\right),$$
$$\phi_{,M}^{*}v \cdot_{\phi_{,M}^{*}TE} \delta_i\Phi_{,M} = \nabla^{\phi^{*}TS}\delta_i\Phi \in \Gamma\left(\phi^{*}TS \otimes_M T^{*}M\right).$$

Proof. This calculation determines the σ component of $\delta_i \Phi_{,M}$.

$$\begin{aligned}
\phi_{,M}^{*}\sigma \cdot_{\phi_{,M}^{*}TE} \delta_i \Phi_{,M} &= \phi_{,M}^{*} \nabla \pi_S \cdot_{\phi_{,M}^{*}TE} \delta_i \Phi_{,M} \\
&= \delta_i \left(\pi_S \circ \Phi_{,M} \right) \\
&= \delta_i \left(\mathrm{Pr}_S^{S\times M} \circ \pi \circ \Phi_{,M} \right) \\
&= \delta_i \Phi \in \Gamma\left(z^{*}\Phi^{*}TS \right) \cong \Gamma\left(\phi^{*}TS \right).
\end{aligned}$$

This calculation determines the μ component of $\delta_i \Phi_{,M}$.

$$\begin{aligned}
\phi_{,M}^{*}\mu \cdot_{\phi_{,M}^{*}TE} \delta_i \Phi_{,M} &= \phi_{,M}^{*} \nabla \pi_M \cdot_{\phi_{,M}^{*}TE} \delta_i \Phi_{,M} \\
&= \delta_i \left(\pi_M \circ \Phi_{,M} \right) \\
&= \delta_i \left(\mathrm{Pr}_M^{S\times M} \circ \pi \circ \Phi_{,M} \right) \\
&= \delta_i \, \mathrm{Pr}_M^{M\times I} \\
&= 0 \in \Gamma\left(z^{*} \left(\mathrm{Pr}_M^{M\times I} \right)^{*} TM \right) \cong \Gamma\left(TM \right).
\end{aligned}$$

The last equality follows from the fact that $\mathrm{Pr}_M^{M\times I}$ does not depend on the i coordinate.

This calculation determines the v component of $\delta_i \Phi_{,M}$. Let $p_M := \mathrm{Pr}_M^{M\times I}$ and $p_I := \mathrm{Pr}_I^{M\times I}$. The left-hand side of the third equality claimed in the lemma will be examined before evaluating at $i = 0$;

$$\begin{aligned}
\Phi_{,M}^{*}v \cdot_{\Phi_{,M}^{*}TE} \partial_i \Phi_{,M} &= \nabla_{p_I^{*}\partial_i}^{\left(\pi\circ\Phi_{,M}\right)^{*}E} \Phi_{,M} \\
&= \nabla_{p_I^{*}\partial_i}^{\left(\Phi\times_{M\times I} p_M\right)^{*}\left(TS\otimes T^{*}M\right)} \Phi_{,M} \in \Gamma\left(\Phi^{*}TS \otimes_{M\times I} p_M^{*}T^{*}M \right).
\end{aligned}$$

Let $Y \in \Gamma\left(TM\right)$, noting that $p_M^{*}Y \in \Gamma\left(p_M^{*}TM\right)$. Then

$$\begin{aligned}
&\left(\Phi_{,M}^{*}v \cdot_{\Phi_{,M}^{*}TE} \partial_i \Phi_{,M} \right) \cdot_{p_M^{*}TM} p_M^{*}Y \\
&= \nabla_{p_I^{*}\partial_i}^{\Phi^{*}TS\otimes_{M\times I} p_M^{*}T^{*}M} \Phi_{,M} \cdot_{p_M^{*}TM} p_M^{*}Y \\
&= \left(\Phi_{,MI} \cdot_{p_I^{*}TI} p_I^{*}\partial_i \right) \cdot_{p_M^{*}TM} p_M^{*}Y \\
&= \left(\Phi_{,IM} \cdot_{p_M^{*}TM} p_M^{*}Y \right) \cdot_{p_I^{*}TI} p_I^{*}\partial_i \\
&\quad \text{(by (23.3))} \\
&= \left(\Phi_{,I} \cdot_{p_I^{*}TI} p_I^{*}\partial_i \right)_{,M} \cdot_{p_M^{*}TM} p_M^{*}Y - \Phi_{,I} \cdot_{p_I^{*}TI} \left(\left(p_I^{*}\partial_i\right)_{,M} \cdot_{p_M^{*}TM} p_M^{*}Y \right) \\
&= \left(\partial_i \Phi \right)_{,M} \cdot_{p_M^{*}TM} p_M^{*}Y \\
&\quad \text{(since } p_I^{*}\partial_i \text{doesn't depend on } M\text{).}
\end{aligned}$$

Recall that $\mathrm{Id}_M = p_M \circ z$ and that the pullback of bundles is contravariant. Then evaluating at $i = 0$ via pullback by z renders

$$
\begin{aligned}
&\left(\phi_{,M}^* v \cdot_{\phi_{,M}^* TE} \delta_i \Phi_{,M}\right) \cdot_{TM} Y \\
&= \left((\Phi_{,M} \circ z)^* v \cdot_{(\Phi_{,M} \circ z)^* TE} z^* \partial_i \Phi_{,M}\right) \cdot_{(p_M \circ z)^* TM} (p_M \circ z)^* Y \\
&= \left(z^* \Phi_{,M}^* v \cdot_{z^* \Phi_{,M}^* TE} z^* \partial_i \Phi_{,M}\right) \cdot_{z^* p_M^* TM} z^* p_M^* Y \\
&= z^* \left(\left(\Phi_{,M}^* v \cdot_{\Phi_{,M}^* TE} \partial_i \Phi_{,M}\right) \cdot_{p_M^* TM} p_M^* Y\right) \\
&= z^* \left((\partial_i \Phi)_{,M} \cdot_{p_M^* TM} p_M^* Y\right) \\
&= z^* (\partial_i \Phi)_{,M} \cdot_{z^* p_M^* TM} z^* p_M^* Y \\
&= (z^* \partial_i \Phi)_{,M} \cdot_{(p_M \circ z)^* TM} (p_M \circ z)^* Y \\
&\quad \text{(by (23.6))} \\
&= (\delta_i \Phi)_{,M} \cdot_{TM} Y \\
&= \nabla^{\phi^* TS} \delta_i \Phi \cdot_{TM} Y.
\end{aligned}
$$

The last equality is because $\delta_i \Phi \in \Gamma(\phi^* TS)$, which is a bundle over M, and therefore $(\delta_i \Phi)_{,M}$ is the total covariant derivative. Because Y is pointwise-arbitrary in TM, this implies that $\phi_{,M}^* \cdot_{\phi_{,M}^* TI} \delta_i \Phi_{,M} = \nabla^{\phi^* TS} \delta_i \Phi$, i.e. the variational derivative δ_i commutes with the first material derivative, just as in the analogous situation in elementary calculus of variations. □

Corollary 25.4 (Euler-Lagrange equations)**.** *If $\phi \in C^\infty(M, S)$ is a critical point of $\mathcal{L}$ (i.e. if $d\mathcal{L}(\phi) \cdot A = 0$ for all $A \in \Gamma(\phi^* TS)$), then*

$$
\begin{aligned}
\phi_{,M}^* L_{,\sigma} - \mathrm{Div}_M \left(\phi_{,M}^* L_{,v}\right) &= 0 \text{ on } M, \\
\phi_{,M}^* L_{,v} \cdot_{TM} \nu &= 0 \text{ on } \partial M.
\end{aligned}
$$

These are called the ***Euler-Lagrange equations*** *for the energy functional $\mathcal{L}$. Recall that because the domain of ϕ is M, $\boldsymbol{\nabla}\phi \equiv \phi_{,M}$.*

Proof. This follows trivially from (25.1) and the Fundamental Lemma of the Calculus of Variations (see [GH96, pg. 16]). □

It should be noted that the boundary Euler-Lagrange equation is due to the fact that the admissible variations are entirely unrestricted. If, for example, the class of maps being considered had fixed boundary data, then any variation would vanish at the boundary, and there would be no boundary Euler-Lagrange equation; this is typically how geodesics and harmonic maps are formulated.

Remark 25.5 (Analogs in elementary calculus of variations). The quantities $L_{,\mu}, L_{,\sigma}, L_{,v}$ generalize the quantities $\frac{\partial L}{\partial x}, \frac{\partial L}{\partial z}, \frac{\partial L}{\partial p}$ respectively of the elementary treatment of the calculus of variations for energy functional

$$(f\colon U \to \mathbb{R}^n) \mapsto \int_U L\left(x, f\left(x\right), Df\left(x\right)\right) \, dx,$$

where $U \subset \mathbb{R}^m$ is compact and $U \times \mathbb{R}^n \times \mathbb{R}^{m \times n} \ni (x, z, p) \mapsto L\left(x, z, p\right)$ is the Lagrangian. Here, $\frac{\partial L}{\partial x}\colon U \times \mathbb{R}^n \times \mathbb{R}^{m\times n} \to \mathbb{R}^m$, $\frac{\partial L}{\partial z}\colon U \times \mathbb{R}^n \times \mathbb{R}^{m\times n} \to \mathbb{R}^n$, and $\frac{\partial L}{\partial p}\colon U \times \mathbb{R}^n \times \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ decompose the total derivative dL and are defined by the relation

$$dL\left(x, z, p\right) \cdot (u, v, w) = \frac{\partial L}{\partial x}\left(x, z, p\right) \cdot u + \frac{\partial L}{\partial z}\left(x, z, p\right) \cdot v + \frac{\partial L}{\partial p}\left(x, z, p\right) : w$$

for $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$, and $w \in \mathbb{R}^{m\times n}$. The Euler-Lagrange equation in this setting is

$$\left(\frac{\partial L}{\partial z} - \mathrm{Div}_U \frac{\partial L}{\partial p}\right)\left(x, f\left(x\right), Df\left(x\right)\right) = 0 \text{ for } x \in U,$$

noting that the left hand side of the equation takes values in $\mathbb{R}^n$.

In most situations involving simpler calculations, it is desirable and acceptable to dispense with the highly decorated notation and use trimmed-town, context-dependent notation, leaving off type-specifying sub/superscripts when clear from context.

Proposition 25.6 (Conserved quantity). *If M is a real interval, $\phi \in C^\infty\left(M, S\right)$ satisfies the Euler-Lagrange equation, and $L_{,\mu} = 0$, then*

$$H := \left(\boldsymbol{\nabla}\, \phi\right)^* L_{,v} \cdot_{\phi^* TS \otimes_M T^* M} \boldsymbol{\nabla}\, \phi - \left(\boldsymbol{\nabla}\, \phi\right)^* L \in C^\infty\left(M, \mathbb{R}\right)$$

is constant. If L is kinetic minus potential energy, then H is kinetic plus potential energy (the total energy), and is referred to as the Hamiltonian.

Proof. Let t be the standard real coordinate. Note that because M is a real interval, $\boldsymbol{\nabla}\, \phi = \phi' \otimes_M dt$. Terms appearing in the derivative of H can be simplified as follows.

Note the repeated $\boldsymbol{\nabla}$ derivatives; $\boldsymbol{\nabla}\phi\colon M \to \phi^* TS \otimes_M T^*M$ but $\boldsymbol{\nabla}\,\boldsymbol{\nabla}\phi\colon M \to (\boldsymbol{\nabla}\phi)^* T\left(\phi^* TS \otimes_M T^*M\right)$ lands in a higher tangent space.

$$
\begin{aligned}
(\boldsymbol{\nabla}\phi)^*\sigma \cdot \boldsymbol{\nabla}\,\boldsymbol{\nabla}\phi \cdot \frac{d}{dt} &= (\boldsymbol{\nabla}\phi)^*\,\boldsymbol{\nabla}\pi_S \cdot \boldsymbol{\nabla}\,\boldsymbol{\nabla}\phi \cdot \frac{d}{dt} = \frac{d}{dt}\left(\pi_S \circ \boldsymbol{\nabla}\phi\right) = \phi', \\
(\boldsymbol{\nabla}\phi)^* v \cdot \boldsymbol{\nabla}\,\boldsymbol{\nabla}\phi \cdot \frac{d}{dt} &= (\boldsymbol{\nabla}\phi)^* v \cdot \frac{d}{dt}\boldsymbol{\nabla}\phi = \nabla_{\frac{d}{dt}}\boldsymbol{\nabla}\phi, \\
\nabla_{\frac{d}{dt}}(\boldsymbol{\nabla}\phi)^* L &= (\boldsymbol{\nabla}\phi)^*\,\nabla L \cdot \boldsymbol{\nabla}\,\boldsymbol{\nabla}\phi \cdot \frac{d}{dt} \\
&= (\boldsymbol{\nabla}\phi)^* L_{,\sigma} \cdot \phi' + (\boldsymbol{\nabla}\phi)^* L_{,v} \cdot \nabla_{\frac{d}{dt}}\boldsymbol{\nabla}\phi, \\
\nabla_{\frac{d}{dt}}\left((\boldsymbol{\nabla}\phi)^* L_{,v} : \boldsymbol{\nabla}\phi\right) &= \nabla_{\frac{d}{dt}}(\boldsymbol{\nabla}\phi)^* L_{,v} : \left(\phi' \otimes_M dt\right) + (\boldsymbol{\nabla}\phi)^* L_{,v} : \nabla_{\frac{d}{dt}}\boldsymbol{\nabla}\phi \\
&= \left(\nabla_{\frac{d}{dt}}(\boldsymbol{\nabla}\phi)^* L_{,v} \cdot dt\right) \cdot \phi' + (\boldsymbol{\nabla}\phi)^* L_{,v} : \nabla_{\frac{d}{dt}}\boldsymbol{\nabla}\phi.
\end{aligned}
$$

Again, because M is a real interval, the divergence is just the derivative, so the Euler-Lagrange equation is

$$
\begin{aligned}
0 &= (\boldsymbol{\nabla}\phi)^* L_{,\sigma} - \operatorname{Div}_M\left((\boldsymbol{\nabla}\phi)^* L_{,v}\right) \\
&= (\boldsymbol{\nabla}\phi)^* L_{,\sigma} - \nabla(\boldsymbol{\nabla}\phi)^* L_{,v} : \left(dt \otimes_M \frac{d}{dt}\right) \\
&= (\boldsymbol{\nabla}\phi)^* L_{,\sigma} - \nabla_{\frac{d}{dt}}(\boldsymbol{\nabla}\phi)^* L_{,v} \cdot dt,
\end{aligned}
$$

and therefore $\nabla_{\frac{d}{dt}}(\boldsymbol{\nabla}\phi)^* L_{,v} \cdot dt = (\boldsymbol{\nabla}\phi)^* L_{,\sigma}$. Thus

$$
\nabla_{\frac{d}{dt}} H = \nabla_{\frac{d}{dt}}\left((\boldsymbol{\nabla}\phi)^* L_{,v} : \boldsymbol{\nabla}\phi - (\boldsymbol{\nabla}\phi)^* L\right) = \left(\nabla_{\frac{d}{dt}}(\boldsymbol{\nabla}\phi)^* L_{,v} \cdot dt - (\boldsymbol{\nabla}\phi)^* L_{,\sigma}\right) \cdot \phi'
$$

which is zero because ϕ satisfies the Euler-Lagrange equation. This shows that H is constant along solutions of the Euler-Lagrange equation, and is therefore a conserved quantity. It should be noted that this proof relies on the fact that the divergence takes a particularly simple form when the domain M is a real interval; the result does not necessarily hold for a general choice of M. □

Example 25.7 (Harmonic maps)**.** Define a metric

$$
k \in \Gamma\left(E^* \otimes_{S\times M} E^*\right) \cong \Gamma\left((TS \otimes T^*M) \otimes_{S\times M} (TS \otimes T^*M)\right)
$$

in a manner analogous to that in (16.4);

$$
k := h \boxtimes g^{-1}.
$$

To clarify, $h \otimes g^{-1} \in \Gamma\left(\left(T^{*}S \otimes_{S} T^{*}S\right) \otimes \left(TM \otimes_{M} TM\right)\right)$, so permuting the middle two components (as in the definition of $h \boxtimes g^{-1}$) gives the correct type, including the necessary metric symmetry condition. If $A \in E$, then $|A|_{k}^{2}$ is the quantity obtained by raising/lowering the indices of A and pairing it naturally with A. A useful fact is that $\nabla k = 0$; if $u \oplus v \in TS \oplus TM$, then permutation commutativity (21.12) and the product rule gives

$$\nabla_{u \oplus v} k = \nabla_{u \oplus v}\left(h \boxtimes g^{-1}\right) = \nabla_{u} h \boxtimes g^{-1} + h \boxtimes \nabla_{v} g^{-1},$$

which equals zero because h and g^{-1} are parallel with respect to ∇^{TS} and $\nabla^{T^{*}M}$ respectively.

With Lagrangian

$$L \colon E \to \mathbb{R},\ A \mapsto \frac{1}{2}|A|_{k}^{2}$$

and energy functional

$$\mathcal{E}(\phi) := \int_{M} L \circ \nabla\!\!\!\!\nabla\, \phi \, dV_{g}$$

($\mathcal{E}(\phi)$ is called the **energy** of ϕ), the resulting Euler-Lagrange equations can be written down after calculating $L_{,\sigma}$ and $L_{,v}$. It is worthwhile to note that L is a quadratic form $A \mapsto A : \frac{1}{2}k : A$ on E, which will automatically imply that $L_{,v}(A) = A : k$. However, the calculation showing this will be carried out for demonstration purposes.

Let $A, B \in TS \otimes T^{*}M$. Then $\epsilon \mapsto A + \epsilon B$ is a vertical variation of A, since $h\left(\delta\left(A + \epsilon B\right)\right) = 0$, so

$$\begin{aligned}
L_{,v}(A) : B &= L_{,v}(A) : v \cdot \delta_{\epsilon}\left(A + \epsilon B\right) \\
&= \delta_{\epsilon}\left(L\left(A + \epsilon B\right)\right) \\
&= \delta_{\epsilon}\left(\left(A + \epsilon B\right) : \frac{1}{2}\left(\pi^{*}k\left(A + \epsilon B\right)\right) : \left(A + \epsilon B\right)\right).
\end{aligned}$$

The product rule gives three terms. The middle term is zero because $\pi\left(A + \epsilon B\right) = \pi(A)$, and therefore does not depend on ϵ. The basepoint evaluation notation for $\pi^{*}k(A)$ will be suppressed for brevity (see Section 18). Thus

$$L_{,v}(A) : B = B : \frac{1}{2}k : A + A : \frac{1}{2}k : B = A : k : B,$$

where the last equality results from the symmetry of k. By the nondegeneracy of the natural pairing on $TS \otimes T^{*}M$ (which is denoted here by :), this implies that $L_{,v}(A) = A : k$.

To calculate $L_{,\sigma}$, it is sufficient (and can be easier) to calculate $L_{,h}$, as $h = \nabla\!\!\!\!\nabla\, \pi$, $\pi = \pi_S \times_E \pi_M$, so $h = \sigma \oplus_E \mu$. Let $A(\epsilon)$ be a horizontal curve in $E = TS \otimes T^*M$; this means that $v \cdot \frac{d}{d\epsilon} A = 0$. Recall that $v \cdot \frac{d}{d\epsilon} A$ is defined by $\nabla^{(\pi \circ A)^* E}_{\frac{d}{d\epsilon}} A$. Then

$$\begin{aligned} L_{,h}(A) \cdot_{\pi^*(TS \oplus TM)} h \cdot_{TE} \delta_\epsilon A &= \big(L_{,h} \cdot_{\pi^*(TS \oplus TM)} h + L_{,v} \cdot_{\pi^* E} v \big) \cdot_{TE} \delta_\epsilon A \\ &= \nabla L \cdot_{TE} \delta_\epsilon A \\ &= \delta_\epsilon (L \circ A) \\ &= \delta_\epsilon \left(A : \frac{1}{2} A^* \pi^* k : A \right). \end{aligned}$$

As before, the product rule gives three terms. Using the contravariance of bundle pullback, the middle term is

$$\frac{1}{2} \nabla^{(\pi \circ A)^* (E^* \otimes_{S \times M} E^*)}_{\delta_\epsilon} (\pi \circ A)^* k = \frac{1}{2} (\pi \circ A)^* \nabla^{E^* \otimes_{S \times M} E^*} k \cdot \delta_\epsilon (\pi \circ A),$$

which equals zero because $\nabla k = 0$. Thus

$$L_{,h}(A) \cdot h \cdot \delta_\epsilon A = \nabla_{\delta_\epsilon} A : \frac{1}{2} k : A + A : \frac{1}{2} k : \nabla_{\delta_\epsilon} A,$$

which equals zero because $\nabla_{\delta_\epsilon} A = v \cdot \delta_\epsilon A = 0$. The quantity $h \cdot \delta_\epsilon A$ can take any value in $\pi^*(TS \oplus TM)$, showing that $L_{,h} = 0$. Finally, $h = \sigma \oplus_E \mu$ implies that $L_{,\sigma} = 0$ and $L_{,\mu} = 0$. This can be understood from the fact that L depends only on the fiber values of A, and has no explicit dependence on the basepoint; this relies crucially on the fact that $\nabla k = 0$.

Finally, the Euler-Lagrange equations can be written down. Recalling that the natural trace of a tensor (used in the divergence term in the Euler-Lagrange equation) is contraction with the appropriate identity tensor, let (e_i) be a local frame for TM and let (e^i) be its dual coframe, so that $e_i \otimes_M e^i$ is a local expression[22] for $\mathrm{Id}_{TM} \in \Gamma(TM \otimes_M T^*M)$. The type-subscripted notation will be minimized except to help clarify. On M:

$$\begin{aligned} 0 &= (\nabla\!\!\!\!\nabla\, \phi)^* L_{,\sigma} - \mathrm{Div}_M ((\nabla\!\!\!\!\nabla\, \phi)^* L_{,v}) \\ &= -\operatorname{Tr} \nabla (\nabla\!\!\!\!\nabla\, \phi : k) \\ &= -\nabla_{e_i} (\nabla\!\!\!\!\nabla\, \phi : k) \cdot_{T^*M} e^i \\ &= -\nabla_{e_i} \nabla\!\!\!\!\nabla\, \phi : k \cdot e^i - \nabla\!\!\!\!\nabla\, \phi : \nabla_{e_i} k \cdot e^i. \end{aligned}$$

[22]It should be noted that while Id_{TM} is being written as the local expression $e_i \otimes_M e^i$, no inherently local property is being used; this tensor decomposition is only used so that the product rule can be used in the following calculations in a clear way.

The second term vanishes because $\nabla k = 0$. Unraveling the definition of k gives $\nabla_{e_i} \overline{\nabla} \phi : k = \phi^* h \cdot \nabla_{e_i} \overline{\nabla} \phi \cdot g^{-1}$. Contracting both sides of the above equation with $-\phi^* h^{-1}$ gives

$$0 = \nabla_{e_i} \overline{\nabla} \phi \cdot g^{-1} \cdot e^i = \nabla_{e_i} \overline{\nabla} \phi \cdot e_i = \operatorname{Tr}_g \nabla^2 \phi \in \Gamma(\phi^* TS).$$

The quantity $\operatorname{Tr}_g \nabla^2 \phi$ is the g-trace of the covariant Hessian of ϕ and can rightfully be called the **covariant Laplacian** of ϕ and denoted by $\Delta_g \phi$ (this is also referred to as the **tension field** of ϕ in other literature (see [Xin96, pg. 13]), which is denoted $\tau(\phi)$). Note that $\Delta_g \phi$ is a vector field along ϕ. This makes sense because ϕ is not necessarily a scalar function; it takes values in S. In the case $S = \mathbb{R}$, $\Delta_g \phi$ is the ordinary covariant Laplacian on scalar functions.

A **harmonic map** is defined as a critical point of the energy functional $\mathcal{E}(\phi) := \int_M \frac{1}{2} |\overline{\nabla} \phi|_k^2 \, dV_M$. Assuming a fixed boundary (so that the variations vanish on the boundary) eliminates the boundary Euler-Lagrange equation, the remaining equation is

$$\Delta_g \phi = 0 \text{ on the interior of } M,$$

which is the generalization of Laplace's equation. Satisfying Laplace's equation is a sufficient condition for a map to be a critical point of the energy functional. There is an abundance of literature concerning harmonic maps and the analysis thereof (see [Con83, GH96, Nis02, Xin96]).

Example 25.8 (The geodesic equation)**.** A fundamental problem in differential geometry is determining length-minimizing curves between given points. If M is a bounded, real interval, and t denotes the standard real coordinate, then the length functional on curves $\phi\colon M \to S$ is $\mathcal{L}(\phi) := \int_M |\phi'|_g \, dt$. A topological metric $d\colon M \times M \to \mathbb{R}$ on M can be defined as

$$d(p, q) := \inf \{\mathcal{L}(\phi) \mid \phi \text{ joins } p \text{to } q\}.$$

It can be shown that the length functional $\mathcal{L}(\phi) := \int_M |\phi'|_h \, dt$ and the energy functional $\mathcal{E}(\phi) := \int_M \frac{1}{2} |\phi'|_h^2 \, dt$ have identical minimizers. Note that $\phi' \in \Gamma(\phi^* TS)$. It is therefore sufficient to consider the analytically preferable energy functional.

In this case, the metric g on M is just scalar multiplication on $\mathbb{R}$. Because M is one-dimensional and t is the standard real coordinate, $\frac{d}{dt}$ is a global, parallel orthonormal frame for TM, and the g-trace of $\nabla^2 \phi$ (i.e. $\Delta_g \phi$) has a single term. The Euler-Lagrange equation, on the interior of M, is

$$0 = \Delta_g \phi = \operatorname{Tr}_g \nabla^2 \phi = \nabla \overline{\nabla} \phi : \left(\frac{d}{dt}, \frac{d}{dt} \right) = \nabla_{\frac{d}{dt}} \overline{\nabla} \phi \cdot \frac{d}{dt} = \nabla_{\frac{d}{dt}} \left(\overline{\nabla} \phi \cdot \frac{d}{dt} \right) - \overline{\nabla} \phi \cdot \nabla_{\frac{d}{dt}} \frac{d}{dt}.$$

But $\nabla\!\!\!\!\nabla \phi \cdot \frac{d}{dt} = \phi'$ and $\nabla_{\frac{d}{dt}} \frac{d}{dt} = 0$, giving the **geodesic equation**

$$\nabla^{\phi^* TS}_{\frac{d}{dt}} \phi' = 0 \text{ on the interior of } M.$$

This is the covariant way to state that the acceleration of ϕ is identically zero. The geodesic equation is commonly notated as $0 = \nabla_{\phi'} \phi'$, though such notation is inaccurate because ϕ' is not a vector field on S, but a vector field along ϕ, and therefore use of the pullback covariant derivative $\nabla^{\phi^* TS}$ is correct (see (21.8)).

While formulated using fixed boundary conditions (ϕ has p and q as its endpoints), the geodesic equation is a second order ODE for which initial tangent vector conditions are sufficient to uniquely determine a solution.

26 Second Variation

A further consideration after finding critical points of the energy functional $\mathcal{L}$ is determining which critical points are extrema. This will involve calculating the second derivative of $\mathcal{L}$. Let $C := C^\infty(M, S)$, noting that $T_\phi C \cong \Gamma(\phi^* TS)$ for $\phi \in C$. The first derivative of $\mathcal{L}$ is $\nabla^{C \to \mathbb{R}} \mathcal{L} := d\mathcal{L}$, as seen in the previous section. The second derivative is the covariant Hessian $\nabla^{T^*C} \nabla^{C \to \mathbb{R}} \mathcal{L}$, where the covariant derivative ∇^{T^*C} is induced by ∇^{TS} (see [Eli67, Theorem 5.4]).

For the remainder of this section, let $I, J \subseteq \mathbb{R}$ be neighborhoods of zero, let i and j be their respective standard coordinates, and extend the existing δ-style derivative-at-a-point notation by defining $\delta_i := \frac{\partial}{\partial i} |_{i=j=0}$, $\delta_j := \frac{\partial}{\partial j} |_{i=j=0}$, and evaluation map $z \colon M \to M \times I \times J$, $m \mapsto (m, 0, 0)$. Then $\delta_i = z^* \partial_i$ and $\delta_j = z^* \partial_j$; these will be used as in the calculation of the first variation.

Theorem 26.1 (Second variation of $\mathcal{L}$)**.** *Let $\mathcal{L}$, L, σ, μ, v and ν all be defined as above. If $\phi \in C^\infty(M, S)$ is a critical point of $\mathcal{L}$ and $A, B \in T_\phi C \cong \Gamma(\phi^* TS)$, then the covariant Hessian of $\mathcal{L}$ is*

$$\begin{aligned}
&\nabla^2 \mathcal{L}(\phi) :_{T_\phi C} (A \otimes B) \\
= &\int_M A \cdot_{\phi^* T^* S} \phi^*_{,M} L_{,\sigma\sigma} \cdot_{\phi^* TS} B + A \cdot_{\phi^* T^* S} \phi^*_{,M} L_{,\sigma v} \cdot_{\phi^* TS \otimes_M T^* M} \nabla^{\phi^* TS} B \\
&+ \nabla^{\phi^* TS} A \cdot_{\phi^* T^* S \otimes_M TM} \phi^*_{,M} L_{,v\sigma} \cdot_{\phi^* TS} B \\
&+ \nabla^{\phi^* TS} A \cdot_{\phi^* T^* S \otimes_M TM} \phi^*_{,M} L_{,vv} \cdot_{\phi^* TS \otimes_M T^* M} \nabla^{\phi^* TS} B \\
&- A \cdot_{\phi^* T^* S} \left(\phi^*_{,M} L_{,v} \cdot_{\phi^* TS \otimes_M T^* M} \left(\phi^* R^{TS} \cdot_{\phi^* TS} \phi_{,M} \right) \right) \cdot_{\phi^* TS} B \, dV_g.
\end{aligned}$$

This is often called the ***second variation*** *of* $\mathcal{L}$*. Here,* $R^{TS} \in \Gamma(TS \otimes_S T^*S \otimes_S T^*S \otimes_S T^*S)$ *denotes the Riemannian curvature endomorphism tensor for the Levi-Civita connection on* TS*.*

Proof. Let $\Phi \colon M \times I \times J \to S$ be a two-parameter variation such that $\delta_i \Phi = A$ and $\delta_j \Phi = B$ (e.g. $\Phi(m, i, j) := \exp(iA(m) + jB(m))$). The variation Φ can be naturally identified with a variation $\overline{\Phi} \colon I \times J \to C$, $(i, j) \mapsto (m \mapsto \Phi(m, i, j))$ which is more conducive to the use of C as a manifold. The tensor products in the generally infinite-dimensional TC are taken formally. Let $\overline{z} := (0, 0) \in I \times J$.

By (23.4), taking the algebra formally in the case of infinite-rank vector bundles,

$$\nabla^2 \left(\mathcal{L} \circ \overline{\Phi}\right) = \overline{\Phi}^* \nabla^2 \mathcal{L} :_{\overline{\Phi}^* TC} \left(\nabla \overline{\Phi} \boxtimes_{I \times J} \nabla \overline{\Phi}\right) + \overline{\Phi}^* \nabla \mathcal{L} \cdot_{\overline{\Phi}^* TC} \nabla \nabla \overline{\Phi},$$

so

$$\begin{aligned}
&\left(\nabla^2 \mathcal{L} \circ_C \phi\right) :_{T_\phi C} (A \otimes B) \\
&= \left(\nabla^2 \mathcal{L} \circ_C \phi\right) :_{T_\phi C} \left(\delta_i \overline{\Phi} \otimes \delta_j \overline{\Phi}\right) + (\nabla \mathcal{L} \circ_C \phi) \cdot_{T_\phi C} \nabla_{\delta_j}^{\overline{\Phi}^* TC} \partial_i \overline{\Phi} \\
&\quad \text{(since } \nabla \mathcal{L} \circ_C \phi = 0\text{)} \\
&= \left(\nabla^2 \mathcal{L} \circ_C \overline{\Phi} \circ_{I \times J} \overline{z}\right) :_{\overline{z}^* \overline{\Phi}^* TC} \left(\delta_i \overline{\Phi} \otimes \delta_j \overline{\Phi}\right) + \left(\nabla \mathcal{L} \circ_C \overline{\Phi} \circ_{I \times J} \overline{z}\right) \cdot_{\overline{z}^* \overline{\Phi}^* TC} \nabla_{\delta_j}^{\overline{\Phi}^* TC} \partial_i \overline{\Phi} \\
&= \nabla^2 \left(\mathcal{L} \circ_C \overline{\Phi}\right) :_{TI \oplus TJ} (\delta_i \otimes_{I \times J} \delta_j) \\
&\quad \text{(by above)} \\
&= \delta_j \partial_i \left(\mathcal{L} \circ_C \overline{\Phi}\right) \\
&= \int_M \delta_j \partial_i (L \circ \Phi_{,M}) \, dV_g \\
&= \int_M \nabla^2 (L \circ \Phi_{,M}) :_{TM \oplus TI \oplus TJ} (\delta_i \otimes_{M \times I \times J} \delta_j) \, dV_g \\
&= \int_M A \cdot_{\phi^* T^* S} \phi_{,M}^* L_{,\sigma\sigma} \cdot_{\phi^* TS} B + A \cdot_{\phi^* T^* S} \phi_{,M}^* L_{,\sigma v} \cdot_{\phi^* TS \otimes_M T^* M} \nabla^{\phi^* TS} B \\
&\quad + \nabla^{\phi^* TS} A \cdot_{\phi^* T^* S \otimes_M TM} \phi_{,M}^* L_{,v\sigma} \cdot_{\phi^* TS} B \\
&\quad + \nabla^{\phi^* TS} A \cdot_{\phi^* T^* S \otimes_M TM} \phi_{,M}^* L_{,vv} \cdot_{\phi^* TS \otimes_M T^* M} \nabla^{\phi^* TS} B \\
&\quad - A \cdot_{\phi^* T^* S} \left(\phi_{,M}^* L_{,v} \cdot_{\phi^* TS \otimes_M T^* M} \left(\phi^* R^{TS} \cdot_{\phi^* TS} \phi_{,M}\right)\right) \cdot_{\phi^* TS} B \, dV_g \\
&\quad \text{(by Calculation (1)).}
\end{aligned}$$

Supporting calculations follow.

Calculation (1): Abbreviate $\phi_{,M}^{*}L_{,xy}$ by $\mathbf{L}_{,xy}$. By (23.4),

$$
\begin{aligned}
&\nabla^{2}\left(L\circ\Phi_{,M}\right):_{TM\oplus TI\oplus TJ}\left(\delta_{i}\otimes_{M\times I\times J}\delta_{j}\right)\\
=&\left(\left[\Phi_{,M}^{*}\nabla^{2}L:_{\Phi_{,M}^{*}TE}\left(\overline{\nabla}\,\Phi_{,M}\boxtimes_{M\times I\times J}\overline{\nabla}\,\Phi_{,M}\right)+\Phi_{,M}^{*}\nabla L\cdot_{\Phi_{,M}^{*}TE}\nabla\overline{\nabla}\,\Phi_{,M}\right]\circ z\right)\\
&:_{z^{*}(TM\oplus TI\oplus TJ)}\left(\delta_{i}\otimes_{M}\delta_{j}\right)\\
=&\,z^{*}\Phi_{,M}^{*}\nabla^{2}L:_{z^{*}\Phi_{,M}^{*}TE}\left(\delta_{i}\Phi_{,M}\otimes_{M}\delta_{j}\Phi_{,M}\right)+z^{*}\Phi_{,M}^{*}\nabla L\cdot_{z^{*}\Phi_{,M}^{*}TE}\nabla_{\delta_{j}}^{\Phi_{,M}^{*}TE}\partial_{i}\Phi_{,M}\\
&\text{(by Calculation (2))}\\
=&\,\mathbf{L}_{,\sigma\sigma}:_{\phi_{,M}^{*}\pi_{S}^{*}TS}\left(\delta_{i}\Phi\otimes_{M}\delta_{j}\Phi\right)+\mathbf{L}_{,\sigma v}\cdot_{\phi_{,M}^{*}\pi_{S}^{*}TS\otimes_{M}\phi_{,M}^{*}\pi^{*}E}\left(\delta_{i}\Phi\otimes_{M}\nabla^{\phi^{*}TS}\delta_{j}\Phi\right)\\
&+\mathbf{L}_{,v\sigma}\cdot_{\phi_{,M}^{*}\pi^{*}E\otimes_{M}\phi_{,M}^{*}\pi_{S}^{*}TS}\left(\nabla^{\phi^{*}TS}\delta_{i}\Phi\otimes_{M}\delta_{j}\Phi\right)\\
&+\mathbf{L}_{,vv}:_{\phi_{,M}^{*}\pi^{*}E}\left(\nabla^{\phi^{*}TS}\delta_{i}\Phi\otimes_{M}\nabla^{\phi^{*}TS}\delta_{j}\Phi\right)\\
&+\mathbf{L}_{,\sigma}\cdot_{\phi_{,M}^{*}\pi_{S}^{*}TS}\nabla_{\delta_{j}}^{\Phi^{*}TS}\partial_{i}\Phi+\mathbf{L}_{,v}\cdot_{\phi_{,M}^{*}\pi^{*}E}\nabla^{\phi^{*}TS}\nabla_{\delta_{j}}^{\Phi^{*}TS}\partial_{i}\Phi\\
&+\mathbf{L}_{,v}\cdot_{\phi_{,M}^{*}\pi^{*}E}\left(\left(\mathrm{Id}_{\phi^{*}TS}\otimes_{M}\delta_{i}\Phi\right)\cdot_{\phi^{*}TS\otimes_{M}\phi^{*}T^{*}S}\left(\phi^{*}R^{TS}\cdot_{\phi^{*}TS}\delta_{j}\Phi\right)\cdot_{\phi^{*}TS}\phi_{,M}\right)\\
&\text{(by Calculation (3)).}
\end{aligned}
$$

Note that $\nabla_{\delta_{j}}^{\Phi^{*}TS}\partial_{i}\Phi\in\Gamma\left(\phi^{*}TS\right)$, and since ϕ is a critical point of $\mathcal{L}$,

$$
\int_{M}\mathbf{L}_{,\sigma}\cdot_{\phi_{,M}^{*}\pi_{S}^{*}TS}\nabla_{\delta_{j}}^{\Phi^{*}TS}\partial_{i}\Phi+\mathbf{L}_{,v}\cdot_{\phi_{,M}^{*}\pi^{*}E}\nabla^{\phi^{*}TS}\nabla_{\delta_{j}}^{\Phi^{*}TS}\partial_{i}\Phi\,dV_{g}=0.
$$

Thus

$$
\begin{aligned}
&\int_{M}\nabla^{2}\left(L\circ\Phi_{,M}\right):_{TM\oplus TI\oplus TJ}\left(\delta_{i}\otimes_{M\times I\times J}\delta_{j}\right)\,dV_{g}\\
=&\int_{M}\mathbf{L}_{,\sigma\sigma}:_{\phi_{,M}^{*}\pi_{S}^{*}TS}\left(\delta_{i}\Phi\otimes_{M}\delta_{j}\Phi\right)+\mathbf{L}_{,\sigma v}\cdot_{\phi_{,M}^{*}\pi_{S}^{*}TS\otimes_{M}\phi_{,M}^{*}\pi^{*}E}\left(\delta_{i}\Phi\otimes_{M}\nabla^{\phi^{*}TS}\delta_{j}\Phi\right)\\
&+\mathbf{L}_{,v\sigma}\cdot_{\phi_{,M}^{*}\pi^{*}E\otimes_{M}\phi_{,M}^{*}\pi_{S}^{*}TS}\left(\nabla^{\phi^{*}TS}\delta_{i}\Phi\otimes_{M}\delta_{j}\Phi\right)\\
&+\mathbf{L}_{,vv}:_{\phi_{,M}^{*}\pi^{*}E}\left(\nabla^{\phi^{*}TS}\delta_{i}\Phi\otimes_{M}\nabla^{\phi^{*}TS}\delta_{j}\Phi\right)\\
&+\delta_{i}\Phi\cdot_{\phi^{*}T^{*}S}\left(\mathbf{L}_{,v}\cdot_{\phi_{,M}^{*}\pi^{*}E}\left(\left(\phi^{*}R^{TS}\cdot_{\phi^{*}TS}\delta_{j}\Phi\right)\cdot_{\phi^{*}TS}\phi_{,M}\right)\right)\,dV_{g}\\
=&\int_{M}A\cdot_{\phi^{*}T^{*}S}\mathbf{L}_{,\sigma\sigma}\cdot_{\phi^{*}TS}B+A\cdot_{\phi^{*}T^{*}S}\mathbf{L}_{,\sigma v}\cdot_{\phi^{*}TS\otimes_{M}T^{*}M}\nabla^{\phi^{*}TS}B\\
&+\nabla^{\phi^{*}TS}A\cdot_{\phi^{*}T^{*}S\otimes_{M}TM}\mathbf{L}_{,v\sigma}\cdot_{\phi^{*}TS}B\\
&+\nabla^{\phi^{*}TS}A\cdot_{\phi^{*}T^{*}S\otimes_{M}TM}\mathbf{L}_{,vv}\cdot_{\phi^{*}TS\otimes_{M}T^{*}M}\nabla^{\phi^{*}TS}B\\
&-A\cdot_{\phi^{*}T^{*}S}\left(\mathbf{L}_{,v}\cdot_{\phi^{*}TS\otimes_{M}T^{*}M}\left(\phi^{*}R^{TS}\cdot_{\phi^{*}TS}\phi_{,M}\right)\right)\cdot_{\phi^{*}TS}B\,dV_{g}\\
&\text{(by antisymmetry of curvature tensor).}
\end{aligned}
$$

Calculation (2):

$$
\begin{aligned}
& z^{*}\nabla \overline{\nabla}\,\Phi_{,M} :_{z^{*}(TM\oplus TI\oplus TJ)} (\delta_i \otimes_M \delta_j) \\
&= z^{*}\nabla^{\Phi_{,M}^{*}TE\otimes_{M\times I\times J}(T^{*}M\oplus T^{*}I\oplus T^{*}J)}\,\overline{\nabla}\,\Phi_{,M} :_{z^{*}(TM\oplus TI\oplus TJ)} z^{*}\left(\partial_i \otimes_{M\times I\times J} \partial_j\right) \\
&= z^{*}\left(\nabla^{\Phi_{,M}^{*}TE\otimes_{M\times I\times J}(T^{*}M\oplus T^{*}I\oplus T^{*}J)}\,\overline{\nabla}\,\Phi_{,M} :_{TM\oplus TI\oplus TJ} \left(\partial_i \otimes_{M\times I\times J} \partial_j\right)\right) \\
&= z^{*}\left(\nabla_{\partial_j}^{\Phi_{,M}^{*}TE\otimes_{M\times I\times J}(T^{*}M\oplus T^{*}I\oplus T^{*}J)}\,\overline{\nabla}\,\Phi_{,M} \cdot_{TM\oplus TI\oplus TJ} \partial_i\right) \\
&= z^{*}\nabla_{\partial_j}^{\Phi_{,M}^{*}TE}\left(\overline{\nabla}\,\Phi_{,M} \cdot_{TM\oplus TI\oplus TJ} \partial_i\right) \ \ (\text{since } \nabla_{\partial_j}^{TM\oplus TI\oplus TJ}\partial_i = 0) \\
&= \nabla_{\delta_j}^{\Phi_{,M}^{*}TE}\partial_i\Phi_{,M}.
\end{aligned}
$$

Calculation (3): As calculated in the proof of (25.1),

$$
\begin{aligned}
\phi_{,M}^{*}\sigma \cdot_{\phi_{,M}^{*}TE} \delta_i\Phi_{,M} &= \delta_i\Phi \in \Gamma\left(\phi^{*}TS\right), \\
\phi_{,M}^{*}\mu \cdot_{\phi_{,M}^{*}TE} \delta_i\Phi_{,M} &= 0 \in \Gamma\left(TM\right), \\
\phi_{,M}^{*}v \cdot_{\phi_{,M}^{*}TE} \delta_i\Phi_{,M} &= \nabla^{\phi^{*}TS}\delta_i\Phi \in \Gamma\left(\phi^{*}TS \otimes_M T^{*}M\right).
\end{aligned}
$$

Furthermore, letting $P := \mathrm{Pr}_M^{M\times I\times J}$ for brevity and noting that $P \circ z = \mathrm{Id}_M$,

$$
\begin{aligned}
& \phi_{,M}^{*}\sigma \cdot_{\phi_{,M}^{*}TE} \nabla_{\delta_j}^{\Phi_{,M}^{*}TE}\partial_i\Phi_{,M} \\
&= z^{*}\nabla_{\partial_j}^{\Phi_{,M}^{*}\pi_S^{*}TS}\left(\Phi_{,M}^{*}\sigma \cdot_{\Phi_{,M}^{*}TE} \partial_i\Phi_{,M}\right) && (\text{since } \nabla\sigma = 0) \\
&= z^{*}\nabla_{\partial_j}^{\left(\pi_S\circ\Phi_{,M}\right)^{*}TS}\partial_i\Phi && (\text{using calculation from (25.1)}) \\
&= \nabla_{\delta_j}^{\Phi^{*}TS}\partial_i\Phi \in \Gamma\left(z^{*}\Phi^{*}TS\right) \cong \Gamma\left(\phi^{*}TS\right),
\end{aligned}
$$

$$
\begin{aligned}
& \phi_{,M}^{*}\mu \cdot_{\phi_{,M}^{*}TE} \nabla_{\delta_j}^{\Phi_{,M}^{*}TE}\partial_i\Phi_{,M} \\
&= z^{*}\nabla_{\partial_j}^{\Phi_{,M}^{*}\pi_M^{*}TM}\left(\Phi_{,M}^{*}\mu \cdot_{\Phi_{,M}^{*}TE} \partial_i\Phi_{,M}\right) \\
&\quad (\text{since } \nabla\mu = 0) \\
&= z^{*}\nabla_{\partial_j}^{\left(\pi_M\circ\Phi_{,M}\right)^{*}TM}0 \\
&\quad (\text{using calculation from (25.1)}) \\
&= 0 \in \Gamma\left(z^{*}\left(\pi_M\circ\Phi_{,M}\right)^{*}TM\right) \cong \Gamma\left(z^{*}P^{*}TM\right) \cong \Gamma\left(TM\right),
\end{aligned}
$$

$$
\begin{aligned}
&\phi_{,M}^{*} v \cdot_{\phi_{,M}^{*} TE} \nabla_{\delta_j}^{\Phi_{,M}^{*} TE} \partial_i \Phi_{,M} \\
&= z^{*} \nabla_{\partial_j}^{\Phi_{,M}^{*} \pi^{*} E} \left(\Phi_{,M}^{*} v \cdot_{\Phi_{,M}^{*} TE} \partial_i \Phi_{,M} \right) && (\text{since } \nabla v = 0) \\
&= z^{*} \nabla_{\partial_j}^{\left(\pi \circ \Phi_{,M}\right)^{*} E} \left(\partial_i \Phi \right)_{,M} && (\text{using calculation from } (25.1)).
\end{aligned}
$$

Note that

$$
\phi_{,M}^{*} v \in \Gamma \left(\phi_{,M}^{*} \pi^{*} E \otimes_M \phi_{,M}^{*} T^{*} E \right) \cong \Gamma \left(\left(\phi \times_M \mathrm{Id}_M \right)^{*} E \otimes_M \phi_{,M}^{*} T^{*} E \right),
$$

and therefore

$$
\phi_{,M}^{*} v \cdot_{\phi_{,M}^{*} TE} \nabla_{\delta_j}^{\Phi_{,M}^{*} TE} \partial_i \Phi_{,M} \in \Gamma \left(\left(\phi \times_M \mathrm{Id}_M \right)^{*} E \right) \cong \Gamma \left(\phi^{*} TS \otimes_M T^{*} M \right),
$$

so it suffices to examine its natural pairing with TM elements. Let $X \in \Gamma(TM)$, noting that $X = \mathrm{Id}_M^{*} X = z^{*} P^{*} X$ and that $P^{*} X = TP \cdot (X \oplus 0_{TI} \oplus 0_{TJ}) \in \Gamma(P^{*} TM)$. Then

$$
\begin{aligned}
&\left(\phi_{,M}^{*} v \cdot_{\phi_{,M}^{*} TE} \nabla_{\delta_j}^{\Phi_{,M}^{*} TE} \partial_i \Phi_{,M} \right) \cdot_{TM} X \\
&= z^{*} \nabla_{\partial_j}^{\Phi^{*} TS \otimes_{M \times I \times J} P^{*} T^{*} M} \left(\partial_i \Phi \right)_{,M} \cdot_{z^{*} P^{*} TM} z^{*} P^{*} X \\
&= z^{*} \nabla_{\partial_j}^{\Phi^{*} TS} \left(\left(\partial_i \Phi \right)_{,M} \cdot_{P^{*} TM} \bar{\nabla} P \cdot_{TM \oplus TI \oplus TJ} \left(X \oplus 0_{TI} \oplus 0_{TJ} \right) \right) \\
&\quad - z^{*} \left(\left(\partial_i \Phi \right)_{,M} \cdot \nabla_{\partial_j}^{P^{*} TM} \left(\bar{\nabla} P \cdot_{TM \oplus TI \oplus TJ} \cdot \left(X \oplus 0_{TI} \oplus 0_{TJ} \right) \right) \right) \\
&= z^{*} \left(\nabla_{\partial_j}^{\Phi^{*} TS} \nabla_{X \oplus 0_{TI} \oplus 0_{TJ}}^{\Phi^{*} TS} \partial_i \Phi - \left(\partial_i \Phi \right)_{,M} \cdot 0_{P^{*} TM} \right) \\
&= z^{*} \left(\nabla_{X \oplus 0_{TI} \oplus 0_{TJ}}^{\Phi^{*} TS} \nabla_{\partial_j}^{\Phi^{*} TS} \partial_i \Phi + \nabla_{[\partial_j, X \oplus 0_{TI} \oplus 0_{TJ}]}^{\Phi^{*} TS} \partial_i \Phi - R^{\Phi^{*} TS} \left(\partial_j, X \oplus 0_{TI} \oplus 0_{TJ} \right) \partial_i \Phi \right) \\
&= z^{*} \left(\nabla_{X \oplus 0_{TI} \oplus 0_{TJ}}^{\Phi^{*} TS} \nabla_{\partial_j}^{\Phi^{*} TS} \partial_i \Phi + \nabla_{0}^{\Phi^{*} TS} \partial_i \Phi \right) \\
&\quad - z^{*} \left(\left(\mathrm{Id}_{\Phi^{*} TS} \otimes_{M \times I \times J} \partial_i \Phi \right) \cdot_{\Phi^{*} TS \otimes_{M \times I \times J} \Phi^{*} T^{*} S} R^{\Phi^{*} TS} \right. \\
&\quad \left. :_{TM \oplus TI \oplus TJ} \left(\partial_j \otimes_{M \times I \times J} \left(X \oplus 0_{TI} \oplus 0_{TJ} \right) \right) \right) \\
&= \left(\nabla^{\phi^{*} TS} \nabla_{\delta_j}^{\Phi^{*} TS} \partial_i \Phi + \left(\mathrm{Id}_{\phi^{*} TS} \otimes_M \delta_i \Phi \right) \right. \\
&\quad \left. \cdot_{\phi^{*} TS \otimes_M \phi^{*} T^{*} S} \left(\phi^{*} R^{TS} \cdot_{\phi^{*} TS} \delta_j \Phi \right) \cdot_{\phi^{*} TS} \phi_{,M} \right) \cdot_{TM} X,
\end{aligned}
$$

where the last equality follows from Calculations (4) and (5). Because X is pointwise-arbitrary in TM, this shows that

$$
\begin{aligned}
&\phi_{,M}^{*} v \cdot_{\phi_{,M}^{*} TE} \nabla_{\delta_j}^{\Phi_{,M}^{*} TE} \partial_i \Phi_{,M} \\
&= \left(\nabla_{\delta_j}^{\Phi^{*} TS} \partial_i \Phi \right)_{,M} \\
&\quad + \left(\mathrm{Id}_{\phi^{*} TS} \otimes_M \delta_i \Phi \right) \cdot_{\phi^{*} TS \otimes_M \phi^{*} T^{*} S} \left(\phi^{*} R^{TS} \cdot_{\phi^{*} TS} \delta_j \Phi \right) \cdot_{\phi^{*} TS} \phi_{,M}.
\end{aligned}
$$

Calculation (4):

$$z^* \left(\nabla^{\Phi^*TS}_{X \oplus 0_{TI} \oplus 0_{TJ}} \nabla^{\Phi^*TS}_{\partial_j} \partial_i \Phi + \nabla^{\Phi^*TS}_{0} \partial_i \Phi \right)$$

$$= z^* \left(\nabla^{\Phi^*TS}_{\partial_j} \partial_i \Phi \right)_{,M} \cdot_{z^*P^*TM} z^* P^* X$$

$$= \left(\nabla^{\Phi^*TS}_{\delta_j} \partial_i \Phi \right)_{,M} \cdot_{TM} X \text{ (by (23.6))}$$

$$= \nabla^{\phi^*TS} \nabla^{\Phi^*TS}_{\delta_j} \partial_i \Phi \cdot_{TM} X \text{ (because } \nabla^{\Phi^*TS}_{\delta_j} \partial_i \Phi \in \Gamma\left(z^* \Phi^* TS\right) \cong \Gamma\left(\phi^* TS\right)).$$

Calculation (5):

$$- z^* \left(R^{\Phi^*TS} :_{TM \oplus TI \oplus TJ} \left(\partial_j \otimes_{M \times I \times J} \left(X \oplus 0_{TI} \oplus 0_{TJ} \right) \right) \right)$$

$$= z^* \left(R^{\Phi^*TS} :_{TM \oplus TI \oplus TJ} \left(\left(X \oplus 0_{TI} \oplus 0_{TJ} \right) \otimes_{M \times I \times J} \partial_j \right) \right)$$

(antisymmetry of R^{Φ^*TS})

$$= z^* \left(\Phi^* R^{TS} :_{\Phi^*TS} \left(\nabla\!\!\!\!\nabla \Phi \boxtimes_{M \times I \times J} \nabla\!\!\!\!\nabla \Phi \right) :_{TM \oplus TI \oplus TJ} \left(\left(X \oplus 0_{TI} \oplus 0_{TJ} \right) \otimes_{M \times I \times J} \partial_j \right) \right)$$

(by (23.5))

$$= z^* \left(\Phi^* R^{TS} :_{\Phi^*TS} \left(\left(\Phi_{,M} \cdot_{P^*TM} P^* X \right) \otimes_{M \times I \times J} \partial_j \Phi \right) \right)$$

$$= z^* \left(\left(\Phi^* R^{TS} \cdot_{\Phi^*TS} \partial_j \Phi \right) \cdot_{\Phi^*TS} \Phi_{,M} \cdot_{P^*TM} P^* X \right)$$

$$= \left(z^* \Phi^* R^{TS} \cdot_{z^*\Phi^*TS} z^* \partial_j \Phi \right) \cdot_{z^*\Phi^*TS} z^* \Phi_{,M} \cdot_{z^*P^*TM} z^* P^* X$$

$$= \left(\phi^* R^{TS} \cdot_{\phi^*TS} \delta_j \Phi \right) \cdot_{\phi^*TS} \phi_{,M} \cdot_{TM} X.$$

□

Theorem 26.2 (Second variation of $\mathcal{L}$ (alternate form))**.** *Let $\mathcal{L}$, L, σ, μ, v and ν all be* *defined as above. If $\phi \in C^{\infty}(M,S)$ is a critical point of $\mathcal{L}$ and $A, B \in \Gamma(\phi^* TS)$, then*

$$\begin{aligned}
&\nabla^2 \mathcal{L}(\phi) :_{T_\phi C} (A \otimes B) \\
= &\int_M A \cdot_{\phi^* T^* S} \mathbf{L}_{,\sigma\sigma} \cdot_{\phi^* TS} B + A \cdot_{\phi^* T^* S} \mathbf{L}_{,\sigma v} \cdot_{\phi^* TS \otimes_M T^* M} \nabla^{\phi^* TS} B \\
&- A \cdot_{\phi^* T^* S} \operatorname{Div}_M \mathbf{L}_{,v\sigma} \cdot_{\phi^* TS} B - A \cdot_{\phi^* T^* S} \mathbf{L}_{,v\sigma} \cdot_{T^* M \otimes_M \phi^* TS} \left(\nabla^{\phi^* TS} B\right)^{(1\,2)} \\
&- A \cdot_{\phi^* T^* S} \operatorname{Div}_M \mathbf{L}_{,vv} \cdot_{\phi^* TS \otimes_M T^* M} \nabla^{\phi^* TS} B \\
&- A \cdot_{\phi^* T^* S} \mathbf{L}_{,vv} \cdot_{T^* M \otimes_M \phi^* TS \otimes_M T^* M} \left(\nabla^{\phi^* TS \otimes_M T^* M} \nabla^{\phi^* TS} B\right)^{(1\,2\,3)} \\
&- A \cdot_{\phi^* T^* S} \left(\mathbf{L}_{,v} \cdot_{\phi^* TS \otimes_M T^* M} \left(\phi^* R^{TS} \cdot_{\phi^* TS} \phi_{,M}\right)\right) \cdot_{\phi^* TS} B \, dV_g \\
&+ \int_{\partial M} \left(A \cdot_{\phi^* T^* S} \mathbf{L}_{,v\sigma} \cdot_{\phi^* TS} B\right) \cdot_{T^* M} \nu \\
&+ \left(A \cdot_{\phi^* T^* S} \mathbf{L}_{,vv} \cdot_{\phi^* TS \otimes_M T^* M} \nabla^{\phi^* TS} B\right) \cdot_{T^* M} \nu \, d\overline{V}_g
\end{aligned}$$

Proof. This result follows essentially from (26.1) via several instances of integration by parts to express the integrand(s) entirely in terms of A and not its covariant derivatives. Abbreviate $\phi^*_{,M} L_{,xy}$ by $\mathbf{L}_{,xy}$. Then, integrating by parts allows the covariant derivatives of A to be flipped across the natural pairings over $\phi^* TS$.

$$\begin{aligned}
&\int_M \nabla^{\phi^* TS} A \cdot_{\phi^* T^* S \otimes_M TM} \mathbf{L}_{,v\sigma} \cdot_{\phi^* TS} B \, dV_g \\
= &\int_M \operatorname{Tr}_{TM} \left(\left(\nabla^{\phi^* TS} A\right)^{(1\,2)} \cdot_{\phi^* TS} \mathbf{L}_{,v\sigma} \cdot_{\phi^* TS} B\right) dV_g \\
&(TM \text{ trace is taken separately}) \\
= &\int_M \operatorname{Tr}_{TM} \left(\nabla^{TM} \left(A \cdot_{\phi^* T^* S} \mathbf{L}_{,v\sigma} \cdot_{\phi^* TS} B\right)\right) \\
&- \operatorname{Tr}_{TM} \left(A \cdot_{\phi^* T^* S} \nabla^{\phi^* T^* S \otimes_M TM \otimes_M \phi^* TS} \mathbf{L}_{,v\sigma} \cdot_{\phi^* TS} B\right) \\
&- \operatorname{Tr}_{TM} \left(A \cdot_{\phi^* T^* S} \mathbf{L}_{,v\sigma} \cdot_{\phi^* TS} \nabla^{\phi^* TS} B\right) dV_g \\
&(\text{reverse product rule}) \\
= &\int_M -A \cdot_{\phi^* T^* S} \operatorname{Div}_M \mathbf{L}_{,v\sigma} \cdot_{\phi^* TS} B \\
&- A \cdot_{\phi^* T^* S} \mathbf{L}_{,v\sigma} \cdot_{T^* M \otimes_M \phi^* TS} \left(\nabla^{\phi^* TS} B\right)^{(1\,2)} dV_g \\
&+ \int_{\partial M} \left(A \cdot_{\phi^* T^* S} \mathbf{L}_{,v\sigma} \cdot_{\phi^* TS} B\right) \cdot_{T^* M} \nu \, d\overline{V}_g \\
&(\text{definition of divergence, and divergence theorem}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_M \nabla^{\phi^*TS} A \cdot_{\phi^*T^*S\otimes_M TM} \mathbf{L}_{,vv} \cdot_{\phi^*TS\otimes_M T^*M} \nabla^{\phi^*TS} B \, dV_g \\
&= \int_M \operatorname{Tr}_{TM} \left(\left(\nabla^{\phi^*TS} A \right)^{(1\,2)} \cdot_{\phi^*T^*S} \mathbf{L}_{,vv} \cdot_{\phi^*TS\otimes_M T^*M} \nabla^{\phi^*TS} B \right) dV_g \\
&= \int_M \operatorname{Tr}_{TM} \left(\nabla^{TM} \left(A \cdot_{\phi^*T^*S} \mathbf{L}_{,vv} \cdot_{\phi^*TS\otimes_M T^*M} \nabla^{\phi^*TS} B \right) \right) \\
&\quad - \operatorname{Tr}_{TM} \left(A \cdot_{\phi^*T^*S} \nabla^{\phi^*T^*S\otimes_M TM\otimes_M \phi^*T^*S\otimes_M TM} \mathbf{L}_{,vv} \cdot_{\phi^*TS\otimes_M T^*M} \nabla^{\phi^*TS} B \right) \\
&\quad - \operatorname{Tr}_{TM} \left(A \cdot_{\phi^*T^*S} \mathbf{L}_{,vv} \cdot_{\phi^*TS\otimes_M T^*M} \nabla^{\phi^*TS\otimes_M T^*M} \nabla^{\phi^*TS} B \right) dV_g \\
&= \int_M -A \cdot_{\phi^*T^*S} \operatorname{Div}_M \mathbf{L}_{,vv} \cdot_{\phi^*TS\otimes_M T^*M} \nabla^{\phi^*TS} B \\
&\quad - A \cdot_{\phi^*T^*S} \mathbf{L}_{,vv} \cdot_{T^*M\otimes_M \phi^*TS\otimes_M T^*M} \left(\nabla^{\phi^*TS\otimes_M T^*M} \nabla^{\phi^*TS} B \right)^{(1\,2\,3)} dV_g \\
&\quad + \int_{\partial M} \left(A \cdot_{\phi^*T^*S} \mathbf{L}_{,vv} \cdot_{\phi^*TS\otimes_M T^*M} \nabla^{\phi^*TS} B \right) \cdot_{T^*M} \nu \, d\overline{V}_g .
\end{aligned}$$

Together with (26.1), this gives the desired result. □

27 Questions and Future Work

This paper is a first pass at the development of a strongly-typed tensor calculus formalism. The details of its workings are by no means complete or fully polished, and its landscape is riddled with many tempting rabbit holes which would certainly produce useful results upon exploration, but which were out of the scope of a first exposition. Here is a list of some topics which the author considers worthwhile to pursue, and which will likely be the subject of his future work. Hopefully some of these topics will be inspiring to other mathematicians, and ideally will start a conversation on the subject.

- There refinements to be made to the type system used in this paper in order to achieve better error-checking and possibly more insight into the relevant objects. There are still implicit type identifications being done (mostly the canonical identifications between different pullback bundles).
- The calculations done in this paper are not in an optimally polished and refined state. With experience, certain common operations can be identified, abstract computational rules generated for these operations, and the relevant calculations simplified.

- The language of Category Theory can be used to address the implicit/explicit handling of natural type identifications, for example, the identification used in showing the contravariance of bundle pullback; $\psi^* \phi^* F \cong (\phi \circ \psi)^* F$.

- The details of the particular implementation of the pullback bundle $\phi^* F$ as a submanifold of the direct product $M \times F$ are used in this paper, but there is no reason to "open up the box" like this. For most purposes, the categorical definition of pullback bundle suffices; the pullback bundle can be worked exclusively using its projection maps $\pi_M^{\phi^* F}$ and $\rho_F^{\phi^* F}$. In the author's experience (which occurred too late to be incorporated into this paper), using this abstract interface cleans up calculations involving pullback bundles significantly.

- The type system used for any particular problem or calculation can be enriched or simplified to adjust to the level of detail appropriate for the situation. For example, if $\gamma \in C^\infty(\mathbb{R}, M)$, then $\nabla \gamma \in \Gamma(\gamma^* TM \otimes_{\mathbb{R}} T^* \mathbb{R})$, but if t is the standard coordinate on $\mathbb{R}$, then $\nabla \gamma = \gamma' \otimes_{\mathbb{R}} dt$, where $\gamma' \in \Gamma(\gamma^* TM)$ is given by $\nabla \gamma \cdot \frac{d}{dt}$. This "primed" derivative has a simpler type than the total derivative, and would presumably lead to simpler calculations (e.g. in (25.6)). This "primed" derivative could also be used in the derivation of the first and second variations. While this would simplify the type system, it would diversify the notation and make the computational system less regularized. However, some situations may benefit overall from this.

- The notion of strong typing comes from computer programming languages. The human-driven type-checking which is facilitated by the pedantically decorated notation in this paper can be done by computer by implementing the objects and operations of this tensor calculus formalism in a strongly typed language such as Haskell. This would be a step toward automated calculation checking, and could be considered a step toward automated proof checking from the top down (as opposed to from the bottom up, using a system such as the Coq Proof Assistant).

- Is there some sort of completeness result about the calculational tools and type system in this paper? In other words, is it possible to accomplish "everything" in a global, coordinate-free way using a certain set of tools, such as pullback bundles, covariant derivatives, chain rules, permutations, evaluation-by-pullback?

- The alternate form of the second variation (see (26.2)) can be used to form a generalized Jacobi field equation for a particular energy functional. Analysis of this equation and its solutions may give insights analogous to the standard (geodesic-based) Jacobi field equation.

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